Algebraic Semantics for the Logic of Multiple-source Approximation Systems

Md. Aquil Khan and Mohua Banerjee
Department of Mathematics and Statistics, Indian Institute of Technology, Kanpur 208 016, India
email: mdaquil,mohua@iitk.ac.in

Abstract: A *multiple-source approximation system (MSAS)* is a tuple $\mathfrak{F} := (U, \{R_i\}_{i\in N})$, where U is a non-empty set, N an initial segment of the set $\mathbb{N}$ of positive integers, and each $R_i, i \in N$, is an equivalence relation on the domain U. A quantified propositional modal logic LMSAS was defined in [1] in order to study MSAS. In this paper, we will present an algebraic semantics for LMSAS. Syntax of the logic LMSAS is given as follows:

There is a (i) a non-empty countable set Var of variables, (ii) a (possibly empty) countable set Con of constants, (iii) a non-empty countable set PV of propositional variables and (iv) the propositional constants $\top, \bot$. The set T of *terms* of the language is given by $Var \cup Con$. Using the standard Boolean logical connectives $\neg$ (negation) and $\wedge$ (conjunction), a unary modal connective $\langle t\rangle$ (possibility) for each term $t \in T$, and the universal quantifier $\forall$, well-formed formulae (wffs) of $LMSAS$ are defined recursively as: $\top|\bot|p|\neg\alpha|\alpha \wedge \beta|\langle t\rangle\alpha|\forall x\alpha$, where $p \in PV,\ t \in T,\ x \in Var$, and α, β are wffs. The set of all wffs and closed wffs of LMSAS will be denoted by $\mathcal{F}$ and $\overline{\mathcal{F}}$ respectively. For a wff α of $LMSAS$, $Con(\alpha)$ will denote the set of constants used in α.

Let Γ be a set of wffs of $LMSAS$. An interpretation for Γ is given by a triple $\mathcal{M} := (\mu, V, I)$, where $\mu := (U, \{R_i\}_{i\in N})$ is a MSAS, $V : PV \to \mathcal{P}(U)$ and $I : Con(\Gamma) \to N$. An assignment for an interpretation $\mathcal{M}$ is a map $v : Term(\Gamma) \to N$ such that $v(c) = I(c)$, for each $c \in Con(\Gamma)$.

The *satisfiability* in an interpretation $\mathcal{M} := (\mu, V, I)$ of a wff α of Γ, under an assignment v, and at an object w of the domain U, denoted as $\mathcal{M}, v, w \models \alpha$ is defined inductively as follows:

$\mathcal{M}, v, w \models \langle t\rangle\alpha$, if and only if there exists w' in U such that $wR_{v(t)}w'$ and $\mathcal{M}, v, w' \models \alpha$.

$\mathcal{M}, v, w \models \forall x\alpha$, if and only if for every assignment v' x-equivalent to v, $\mathcal{M}, v', w \models \alpha$.

α is *valid*, denoted $\models \alpha$, if and only if $\mathcal{M}, v, w \models \alpha$, for every interpretation $\mathcal{M} := (\mu, V, I)$, assignment v for $\mathcal{M}$ and object w in the domain of μ.

The following sound and complete deductive system for LMSAS was proposed in [1]. t stands for a term in T.

Axiom schema:

(Ax1). All axioms of classical propositional logic.
(Ax2). $\forall x\alpha \to \alpha(t/x)$, where α admits the term t for the variable x.
(Ax3). $\forall x(\alpha \to \beta) \to (\alpha \to \forall x\beta)$, where the variable x is not free in α.
(Ax4). $\forall x[t]\alpha \to [t]\forall x\alpha$, where the term t and variable x are different.
(Ax5). $[t](\alpha \to \beta) \to ([t]\alpha \to [t]\beta)$. (Ax6). $\alpha \to \langle t\rangle\alpha$.
(Ax7). $\alpha \to [t]\langle t\rangle\alpha$. (Ax8). $\langle t\rangle\langle t\rangle\alpha \to \langle t\rangle\alpha$.

Rules of inference:

$$\forall.\ \frac{\alpha}{\forall x\alpha} \qquad MP.\ \frac{\begin{array}{c}\alpha\\ \alpha \to \beta\end{array}}{\beta} \qquad N.\ \frac{\alpha}{[t]\alpha}$$

We note that this is different from both propositional quantification of modal logic, and modal predicate logic.

Next, we present an algebraic semantics for LMSAS. We begin with the following definition.

Definition 1 *A BAOs $\mathfrak{A} := (A, \cap, \sim, 1, f_i)_{i\in\Delta}$ is said to be a* complete BAOs *(CBAOs) if it satisfies the following properties for all $X \subseteq A$:*
(A1) $\bigcap X$ and $\bigcup X$ exists; (A2) $f_k \bigcap X = \bigcap f_k X,\ k \in \Delta$.

In this paper, we are interested only in those (complete) BAOs where $\Delta = \mathbb{N}$ and each f_k satisfies the following additional conditions:

(B1) $f_k a \le f_k f_k a$; (B2) $f_k a \le a$ and (B3) $a \le f_k g_k a,$ where $g_k :=\sim f_k \sim$.

So by a (complete) BAOs, we shall mean a (complete) BAOs $\mathfrak{A} := (A, \cap, \sim, 1, f_i)_{i\in\Delta}$ where $\Delta = \mathbb{N}$ and each $f_k,\ k \in \mathbb{N}$ satisfies the above three additional conditions. The class of all complete BAOs is denoted by $\mathfrak{C}$. We show completeness of LMSAS with respect to $\mathfrak{C}$.

Let $\mathfrak{A} := (A, \cap, \sim, 1, f_i)_{i\in\mathbb{N}}$ be a BAO's. By an *assignment* in $\mathfrak{A}$, we mean a function $\theta : PV \to A$. θ can be extended uniquely, in the standard way, to a meaning function $\tilde{\theta} : \overline{\mathcal{F}} \to A$ where in particular, $\tilde{\theta}([c_i]\alpha) = f_i(\tilde{\theta}(\alpha)),\ i \in \mathbb{N}$ and $\tilde{\theta}(\forall x\alpha) = \bigcap\{\tilde{\theta}(\alpha(c_j/x)) : j \in \mathbb{N}\}$, provided the g.l.b. exists. We define $\tilde{\theta}(\alpha) := \tilde{\theta}(cl(\alpha)),\ \alpha \in \mathcal{F}$.

Definition 2 *A BAO's $\mathfrak{A} := (A, \cap, \sim, 1, f_i)_{i\in\mathbb{N}}$ is said to be a* realization for LMSAS*, if for every assignment $\theta : PV \to A$ the following is satisfied:*
(R1) $\tilde{\theta}(\alpha)$ exists for all $\alpha \in \overline{\mathcal{F}}$; *(R2) $f_k \bigcap_j \tilde{\theta}(\alpha(c_j/x)) = \bigcap_j f_k\tilde{\theta}(\alpha(c_j/x))$, where α has only one free variable x.*

Note that every complete BAOs is a realization for LMSAS.

Let $\mathfrak{A} := (A, \cap, \sim, 1, f_i)_{i\in\mathbb{N}}$ be a BAOs. Then we write $\mathfrak{A} \Vdash \alpha \approx \beta$ if and only if for every assignment $\theta : PV \to A,\ \tilde{\theta}(\alpha) = \tilde{\theta}(\beta)$. We simply write $\mathfrak{C} \Vdash \alpha$ if $\mathfrak{A} \Vdash \alpha \approx \top$ for all $\mathfrak{A} \in \mathfrak{C}$.

The *Lindenbaum algebra* $\mathfrak{L}$ for LMSAS, constructed following the standard technique, turns out to be a realization for LMSAS. As we are not able to show that the Lindenbaum algebra is a CBAOs, we need to do some more work in order to get the completeness theorem with respect to CBAOs. Note that we would achieve our goal if we could embed any LMSAS realization $\mathfrak{A} := (A, \cap, \sim, 1, f_i)_{i\in\mathbb{N}}$ into some complex algebra. At this point one may think of the BAOs consisting of all subsets of the set of all ultra-filters of the BAOs $\mathfrak{A}$, as described in the Jónsson-Tarski Theorem. But the embedding given in this theorem may not preserve infinite joins and meets. This problem can be overcome if we consider the BAOs consisting of all subsets of the set of all Q-filters [2] instead of ultra-filters. Here, Q is a countably infinite collection of subsets of A satisfying certain conditions and the embedding obtained in this case preserves all the infinite joins and meets *in* Q. Since this embedding may not preserve *all* existing joins and meets, question again arises whether even this embedding will be able to give us the desired result. The answer is yes. In fact, for the canonical assignment θ^c which maps $p \in PV$ to its equivalence class, the joins and meets shall be preserved for the sets $Q_\alpha := \{\tilde{\theta^c}(\alpha(c_j/x)) : j \in \mathbb{N}\}$, where α is a wff which has only one free variable x. Thus taking $Q := \{Q_\alpha : \alpha \text{ has the single free variable } x\}$, we get the result.

Theorem 1 *Let $\mathfrak{A} := (A, \cap, \sim, 1, f_i)_{i\in\mathbb{N}}$ be a LMSAS algebra. Then there exists a complex algebra $\mathfrak{F}^+_{\mathfrak{A}}$ and an embedding r of $\mathfrak{A}$ into $\mathfrak{F}^+_{\mathfrak{A}}$ such that*

$$r(\bigcap_j \tilde{\theta^c}(\alpha(c_j/x))) = \bigcap_j r(\tilde{\theta^c}(\alpha(c_j/x))).$$

Let us return to the Lindenbaum algebra $\mathfrak{L}$. Then we have r and $\mathfrak{F}^+_{\mathfrak{L}}$ from Theorem 1, and get

Proposition 1 *Consider the assignment γ in the BAOs $\mathfrak{F}^+_{\mathfrak{L}}$ defined as $\gamma(p) := r([p]),\ p \in PV$. Then $\tilde{\gamma}(\alpha) = r([\alpha])$ for all $\alpha \in \overline{\mathcal{F}}$.*

Proposition 2 (Completeness Theorem) *(1) $\mathfrak{C} \Vdash \alpha$ implies $\vdash \alpha$ for all $\alpha \in \overline{\mathcal{F}}$.*
(2) $\mathfrak{C} \Vdash \alpha$ implies $\vdash \alpha$ for all $\alpha \in \mathcal{F}$.

Future work: We have proved completeness of LMSAS with respect to the class of CBAO's and LMSAS realization. However, an independent algebraic characterization of the LMSAS realization is yet to be obtained.

References

[1] M. A. Khan and M. Banerjee. Formal reasoning with rough sets in multiple-source approximation systems. *Int. J. Approximate Reasoning*, to appear.

[2] Y. Tanaka and H. Ono. Rasiowa-Sikorski lemma and Kripke completeness of predicate and infinitary modal logics. In Z. Zakharyaschev, K. Segerberg, M. de Rijke, and H. Wansing, editors, *Advances in Modal Logic, Volume 2*, pages 401–419. CSLI Publications, 2001.