UNIVERSAL ALGEBRAIC EXTENSIONS OF STONE DUALITIES FOR DISTRIBUTIVE LATTICES AND HEYTING ALGEBRAS

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In this talk, we consider universal algebraic extensions of Stone dualities for distributive lattices ([2]) and for Heyting algebras. Our slogan is: The coincidence of "term functions" and "continuous proper maps" provides a topological duality. Of course, the slogan is just a slogan and inadequate in some respects. We remark that it is sometimes appropriate to replace "continous proper" with another condition.

Stone duality for Boolean algebras states that the category of Boolean algebras and homomorphisms is dually equivalent to that of Boolean spaces (i.e., zero-dimensional compact Hausdorff spaces) and continuous proper maps, where "proper continuous" is equivalent to "continuous" for maps between Boolean spaces (for the definition of properness, see Definition 1). Note that the class of Boolean algebras coincides with the quasi-variety generated by the two-element Boolean algebra $\{0, 1\}$.

In 1971, Hu [1] generalized this duality to that for the quasi-variety generated by any primal algebra, where a finite algebra L is primal iff, for any $n \in \omega$, the set of all n-ary term functions on L coincides with the set of all functions from L^n to L (see [1]). The two-element Boolean algebra $\{0, 1\}$ is a primal algebra as is well known. Stone-Hu duality states that the category of algebras in the quasi-variety generated by a primal algebra and homomorphisms is dually equivalent to that of Boolean spaces and continuous proper maps.

Let L_d denote the discrete space whose underlying set is L. Then, it is clear that L is primal iff, for any $n \in \omega$, the set of all n-ary term functions on L coincides with the set of all continuous proper maps from $(L_d)^n$ to L_d. This fact demonstrates our slogan.

In 1937, Stone [2] showed a duality for distributive lattices, which states that the category of distributive lattices and homomorphisms is dually equivalent to that of coherent spaces and continuous proper maps (see Definition 1). Note that the class of distributive lattices coincides with the quasi-variety generated by the two-element distributive lattice $\{0, 1\}$. Moreover, we have the following fact: For any $n \in \omega$, the set of all n-ary term functions on $\{0, 1\}$ coincides with the set of all continuous proper maps from $(\{0, 1\}_u)^n$ to $\{0, 1\}_u$, where $\{0, 1\}_u$ denotes the topological space $(\{0, 1\}, \mathcal{O})$ with $\mathcal{O} = \{\emptyset, \{1\}, \{0, 1\}\}$. This also demonstrates our slogan.

Recall that Hu replaced the two-element Boolean algebra $\{0, 1\}$ with a primal algebra in Stone duality for Booelan algebras and showed the general duality theorem stated above. In this talk, based on our slogan, we replace the two-element distributive lattice $\{0, 1\}$ with a finite ordered algebra which is "in harmony with the Alexandrov topology" (see Definition 2) in Stone duality for distributive lattices and show a general duality theorem for the quasi-variety generated by any such algebra (Theorem 5), by which we also extend Stone duality for Heyting algebras to a more general case (Corollary 10). In this way, we can confirm the applicability of our slogan.

1. Main Results

Let L be a finite algebra with a join operation and with the greatest element 1 and the least element 0 with respect to the partial ordering induced by the join operation. $\mathrm{Term}_n(L)$ denotes the set of all n-ary term functions on L. Let L_u be the topological space $(L, \mathcal{O}_L)$ equipped with the Alexandrov topology, i.e., $\mathcal{O}_L$ is generated by $\{\uparrow a\ ;\ a \in L\}$, where $\uparrow a = \{x \in L\ ;\ a \leq x\}$. For $n \in \omega$, we equip $(L_u)^n$ with the product topology.

$\mathbb{ISP}(L)$ denotes the class of all isomorphic copies of subalgebras of direct powers of L.

For $A \in \mathbb{ISP}(L)$, $\mathrm{Spec}_L(A)$ denotes the set of all homomorphisms from A to L. Then define $\langle a \rangle = \{v \in \mathrm{Spec}_L(A)\ ;\ v(a) = 1\}$ for $a \in A$. We equip $\mathrm{Spec}_L(A)$ with the topology generated by $\{\langle a \rangle\ ;\ a \in A\}$ for $a \in A$.

Definition 1. *For topolocigal spaces S_1 and S_2, a map $f : S_1 \to S_2$ is proper iff $f^{-1}(O)$ is a compat open subset of S_1 for any compat open subset O of S_2.*

A topological space S is coherent iff S is a compact sober space and the set of all compact open subsets of S forms an open basis and is closed under finite intersections.

For a topological space S, $\mathrm{ContProp}(S)$ denotes the set of all continous proper maps from S to L_u. We can consider $\mathrm{ContProp}(S)$ as an algebra of the same type as L by equipping $\mathrm{ContProp}(S)$ with the operations defined pointwise, i.e., for $n \in \omega$ and an n-ary operation t of L, an operation t of $\mathrm{ContProp}(S)$ is defined by, for $f_1, ..., f_n \in \mathrm{ContProp}(S)$, $(t(f_1, ..., f_n))(x) = t(f_1(x), ..., f_n(x))$ for all $x \in S$.

Definition 2. *L is in harmony with the Alexandrov topology iff, for any $n \in \omega$, the following holds:*

$$\mathrm{Term}_n(L) = \mathrm{ContProp}((L_u)^n).$$

In the below, we assume that L is in harmony with the Alexandrov topology.

Theorem 3. *Let $A \in \mathbb{ISP}(L)$. Then, A is isomorphic to* $\mathrm{ContProp} \circ \mathrm{Spec}_L(A)$.

Theorem 4. *Let S be a coherent space. Then, S is homeomorphic to* $\mathrm{Spec}_L \circ \mathrm{ContProp}(S)$.

Theorem 5. *The category of algebras in $\mathbb{ISP}(L)$ and homomorphisms is dually equivalent to the category of coherent spaces and continuous proper maps.*

In the case that L is the two-element distributive lattice, the above theorem coincides with the original Stone duality theorem for distributive lattices.

Based on the above theorems, we can generalize Stone duality for Heyting algebras. In the below, we additionally assume that L is endowed with a binary operation $*$ and that L is residuated with respect to $*$ in the following sense:

Definition 6. *A subalgebra A of a direct power of L is residuated with respect to $*$ iff for any $a, b \in A$, there is $c \in A$ such that, for any $x \in A$, $a * x \leq b$ iff $x \leq c$. We denote such c by $a \to b$.*

$\mathbb{IRSP}(L)$ denotes the class of all isomorphic copies of residuated (w.r.t. $*$) subalgebras of direct powers of L. If $L = (\{0, 1\}, \wedge, \vee)$ and $* = \wedge$, then $\mathbb{IRSP}(L)$ coincides with the class of Heyting algebras.

Definition 7. *A topological space S is an L-Heyting space iff S is coherent and* $\mathrm{ContProp}(S)$ *is residuated with respect to $*$.*

By Theorem 5, we have the following:

Corollary 8. *Let $A \in \mathbb{IRSP}(L)$. Then, $(A, \to)$ is isomorphic to* $(\mathrm{ContProp} \circ \mathrm{Spec}_L(A), \to)$.

Corollary 9. *Let S be an L-Heyting space. Then, S is homeomorphic to* $\mathrm{Spec}_L \circ \mathrm{ContProp}(S)$.

Corollary 10. *The category of algebras in $\mathbb{IRSP}(L)$ and homomorphisms is dually equivalent to the category of L-Heyting spaces and continuous proper maps.*

References

[1] T. K. Hu, On the topological duality for primal algebra theory, *Algebra Universalis* 1 (1971), 152-154.

[2] M. H. Stone, Topological representation of distributive lattices and Brouwerian logic, *Casopis pest. Mat. a Fys.* 67 (1937), 1-25.