Relation algebras

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Thanks to the organisers for inviting us!
And Happy New Year!
Workshop outline

1. Introduction to relation algebras

2. Games

3. Monk algebras: completions, canonicity

4. Rainbow construction: non-finite axiomatisability and non-elementary results
This talk: introduction to relation algebras

1. Algebras of relations
   - boolean algebras, relation algebras, representations
   - applications

2. Duality
   - atom structures, representations of atomic relation algebras
   - examples: point algebra, McKenzie & anti-Monk algebras
   - ultrafilters, canonical extensions
   - completions

3. Conclusion; some references
1. Algebras of relations

*Boole:* started the algebraic formalisation of unary relations.
Boolean algebras

– algebras \((B, +, -, 0, 1)\) satisfying these equations, for all \(a, b, c \in B\):

1. \((a + b) + c = a + (b + c)\)
2. \(a + b = b + a\)
3. \(a + a = a\)
4. \(-(-a) = a\)
5. \(a + (-a) = 1\)
6. \(-1 = 0\)
7. \(a \cdot (b + c) = a \cdot b + a \cdot c\), where \(a \cdot b\) abbreviates \(-(-a + -b)\)
8. \(0 + a = a\)

We let \(a \leq b\) abbreviate \(a + b = b\), and \(a < b\) abbreviate \(a \leq b \land a \neq b\).

For \(S \subseteq B\), \(\sum S\) is least upper bound of \(S\) in \(B\), if exists. \(\prod S\): glb.

We sometimes use \(-\) as a binary operator: \(a - b = a \cdot (-b)\).
Boolean algebras and unary relations

Definition: for any set $X$,

- a unary relation on $X$ is just a subset of $X$,

- the algebra of all unary relations on $X$ is $(\mathcal{P}(X), \cup, \neg, \emptyset, X)$. The operations are the ‘natural’ ones.

- an algebra of unary relations on $X$ is any subalgebra of this. Such algebras are also known as fields of sets.

These algebras are boolean algebras (exercise).

Conversely, any boolean algebra is isomorphic to an algebra of unary relations on some set (Stone, 1936).

So boolean algebra axioms are sound and complete for unary relations: every boolean algebra is isomorphic to a field of sets.
De Morgan

(born 27 June 1806 in Madura, Madras Presidency — now Madurai, Tamil Nadu)

— should consider binary (and higher-arity) relations.
Binary relations?

• A *binary relation* on a set $X$ is a subset of $X \times X$.
• Egs: graphs, orderings, equivalence relations. Very important.
• An *algebra of binary relations on $X$* is a subalgebra of

$$\mathcal{R}_e(X) = (\emptyset(X \times X), \cup, \setminus, \emptyset, X \times X, Id_X, -^{-1}, |),$$

where

$$Id_X = \{(x, x): x \in X\},$$

$$R^{-1} = \{(y, x): (x, y) \in R\},$$

$$R | S = \{(x, y): \exists z((x, z) \in R \land (z, y) \in S)\}.$$

This choice of relational operations can be disputed. It does not lead to such a nice picture as for boolean algebras.
Example: family relations

Let $X$ be the set of all people (alive or dead).

Consider the binary relations $son$, $daughter$ on $X$:

$(x, y) \in son$ iff $y$ is a son of $x$, etc. Then

- child = $son \cup daughter$
- grandson = child $|$ $son$
- granddaughter = child $|$ $daughter$
- parent = $child^{-1}$
- sister = $(parent \mid daughter) \cap Id_X$
- aunt = $parent \mid sister$
- mother = $parent \mid ((parent \mid daughter) \cap Id_X)$

Exercise: try to do $sibling$, $niece$, $cousin$. 
Next developments

Peirce and *Schröder* established many properties of binary relations.

But no end in sight…
C. S. Peirce

The logic of relatives is highly multiform; it is characterized by innumerable immediate conclusions from the same set of premises. . . . The effect of these peculiarities is that this algebra cannot be subjected to hard and fast rules like those of the Boolian calculus; and all that can be done in this place is to give a general idea of the way of working with it.
— tried to reformulate Schröder’s results with modern algebra.

He wanted to axiomatise the \textit{algebras of binary relations}.
Relation algebras

In 1940s, Tarski proposed axioms.

They define the class \( \text{RA} \) of ‘relation algebras’: algebras \( \mathcal{A} = (A, +, −, 0, 1, 1', \circ, ;) \) such that

- \( (A, +, −, 0, 1) \) is a boolean algebra
- \( (A, ;, 1') \) is a monoid (a semigroup with identity, 1’)
- Peircean law: \( (a; b) \cdot c \neq 0 \iff (\tilde{a}; c) \cdot b \neq 0 \iff a \cdot (c; \tilde{b}) \neq 0 \) for all \( a, b, c \in A \).

Remark: Tarski’s original axioms were equations. They capture all true equations about relations that can be proved with 4 variables.
Representable relation algebras — RRA

An algebra $\mathcal{A} = (A, +, -, 0, 1, 1', \cdot, ;)$ is said to be representable if it is isomorphic to a subalgebra of a product of algebras of the form $\mathcal{Re}(X)$.

That is, there is an embedding

$$h : \mathcal{A} \rightarrow \prod_{i \in I} \mathcal{Re}(X_i),$$

for some sets $I$ and $X_i$ ($i \in I$).

Such an embedding is called a representation of $\mathcal{A}$.

The base set of $h$ is $\bigcup_{i \in I} X_i$ (the $X_i$ can be assumed disjoint).

The class of representable algebras is denoted $\text{RRA}$. $\text{RRA}$ stands for ‘representable relation algebras’. Easy: $\text{RRA} \subseteq \text{RA}$. $\text{RRA}$ is our main object of study in these talks.
Why products?

An algebra of relations is a subalgebra of some $\mathcal{R}_e(X)$.

So why are the representable algebras defined as those isomorphic to subalgebras of \textit{products} of $\mathcal{R}_e(X)$s?

Likely answer: Tarski wanted the representable algebras to form a \textit{variety} (equationally axiomatised class). So we get a closer notion to boolean algebras (also defined by equations).

Varieties are closed under products. Tarski proved (1955) that RRA, defined with products as above, is a variety.

\textbf{Remark:} a relation algebra $\mathcal{A}$ is \textit{simple} if it has no nontrivial proper homomorphic images. This holds iff $\mathcal{A} \models \forall x(x > 0 \rightarrow 1 ; x ; 1 = 1)$.

Any simple $\mathcal{A} \in \text{RRA}$ has a representation $h : \mathcal{A} \rightarrow \mathcal{R}_e(X)$.

So for \textit{simple} relation algebras, we can relax about products.
Did Tarski’s axioms capture RRA?

Soundness \((\text{RRA} \subseteq \text{RA})\) is easy.

Completeness fails.

Lyndon (1950): gave example of \(\mathcal{A} \in \text{RA} \setminus \text{RRA}\). \((|\mathcal{A}| = 2^{56})\)

Monk (1964): \text{RRA} is not finitely axiomatisable.
More facts about RRA

Many ‘negative’ results about RRA are now known.

- **RRA** cannot be axiomatised by any set of equations using finitely many variables (Jónsson 1988, but Tarski knew in 1975)
- Andréka has results on numbers of occurrences of operations in axioms for representable cylindric algebras (analogous to RRA)
- there is no algorithm to decide whether a finite relation algebra is representable (Hirsch–IH 1999)
- **RRA** is not closed under (Monk) completions (IH 1997)
- there is no Sahlqvist or even canonical axiomatisation of **RRA** (IH–Venema 1997, 2003)

Others for related classes coming later…

*Compare the situation for boolean algebras…!*
Problems

1. Find *simple intrinsic* characterisation of (algebras in) **RRA**. The next talk (games) contributes to this.

2. **Finitization problem:** find expressive operations on binary (and higher-arity) relations, yielding a finitely axiomatisable class of representable algebras.

This is open. Sain, Sayed Ahmed and others have made progress on infinite-arity relations (mainly without equality).

A positive solution could contribute to a finitely axiomatisable algebraisation of first-order logic. The Boolean paradise would be regained.
Applications of relation algebras

Binary relations are fundamental. Results about them, and proof methods, will have applications elsewhere.

1. **Artificial planning**: Allen, Ladkin, Maddux, Hirsch
2. **Databases**: use similar relational operations
3. **Modal logic**: product logics between $K^n$ and $S5^n$ ($n \geq 3$) are undecidable, non-finitely axiomatisable, no algorithm to decide if a finite frame validates the logic (Hirsch–IH–Kurucz 2002)
5. **Temporal logic**: interval logics with Chop and the like are not finitely axiomatisable (IH–Montanari–Sciavicco 2007)
2. Duality for relation algebras

Our aim in these talks is to study relation algebras and sketch proofs of some key results.

*Duality* is often helpful.

It is like *taking logs* in arithmetic — makes life easier.

But the main problems involved in finding representations of relation algebras are hardly touched by duality.

In fact relation algebras shed as much light on duality as vice versa!

We will look at

- atoms, atomic relation algebras, atom structures
- ultrafilters, complete representations, canonical extensions
- completions
2.1 Duality for algebras, by atoms

A relation algebra is a boolean algebra with extra operations. So we can define $a \cdot b$, $a \leq b$, $\sum S$, etc., as in boolean algebras.

An **atom** of a relation algebra (or boolean algebra) is a $\leq$-minimal non-zero element $x$: it satisfies $\forall y(y < x \leftrightarrow y = 0)$.

A relation algebra $\mathcal{A} = (A, +, -, 0, 1, 1', \sim, ;)$ is **atomic** if every non-zero element of $\mathcal{A}$ has an atom beneath ($\leq$) it.

- For any $X$, $\mathcal{R}e(X)$ is atomic.
- Any finite relation algebra is atomic (exercise).
- There are infinite atomless relation algebras.

In an atomic relation algebra $\mathcal{A}$, every element $a$ of $\mathcal{A}$ is the sum of the atoms beneath it:

$$a = \sum \{x : x \text{ an atom of } \mathcal{A}, x \leq a\}.$$


**Completely additive functions**

Let \((B, +, -, 0, 1)\) be a boolean algebra.

A function \(f : B^n \to B\) is **completely additive** if for every \(i \leq n\), \(b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n \in B\), and \(S \subseteq B\) such that \(\sum S\) exists,

\[
f(b_1, \ldots, b_{i-1}, \sum S, b_{i+1}, \ldots, b_n) = \sum \{f(b_1, \ldots, b_{i-1}, s, b_{i+1}, \ldots, b_n) : s \in S\}.
\]

The RA axioms show that \(\sim\) and \(;\) are completely additive: if \(a_i, b_j\) are elements of any relation algebra, then whenever the blue sums exist,

\[
\left(\sum_i a_i\right) \sim = \sum_i \tilde{a}_i \\
\left(\sum_i a_i\right); \left(\sum_j b_j\right) = \sum_{i,j} (a_i; b_j).
\]
Atomic relation algebras and atom structures

We can now specify atomic relation algebras quite easily.

We saw that $\vee$ and $;$ are completely additive. So in an atomic relation algebra $\mathcal{A}$, they are determined by their values on atoms.

(Every element of $\mathcal{A}$ is the sum of the atoms beneath it. So by complete additivity, $a ; b = \sum \{ x ; y : x, y \text{ atoms}, x \leq a, y \leq b \}$.)

So given its boolean part, $\mathcal{A}$ is determined by its atom structure $\text{At } \mathcal{A}$:

- the set of atoms of $\mathcal{A}$,
- which atoms are $\leq 1'$,
- $\check{x}$, for each atom $x$ (turns out that $\check{x}$ is also an atom),
- for each atoms $x, y, z$, whether $x ; y \geq z$. (This determines $x ; y$.)

If $x ; y \geq z$, we say that $(x, y, z)$ is a consistent triple of atoms.

Remark: $(x, y, z)$ is consistent iff its Peircean transforms $(x, y, z), (\check{x}, z, y), (z, \check{y}, x), (y, \check{z}, \check{x}), (\check{z}, x, \check{y}), (\check{y}, \check{x}, \check{z})$ are all consistent.
Examples: two finite relation algebras

1. **McKenzie’s algebra** $\mathcal{K}$.
   
   4 atoms: 1’, <, >, ♯.
   
   $1' = 1'$, $\leq = >$, $> = <$, $\# = \#$.
   
   All triples are consistent except Peircean transforms of:
   $(1', a, a')$ for $a \neq a'$, $(<, <, >)$, $(<, <, \#)$, $(\#, \#, \#)$.

2. **The ‘anti-Monk algebra’ $\mathcal{M}$ (Maddux?)**
   
   4 atoms: 1’, $r$, $b$, $g$.
   
   $\tilde{a} = a$ for all atoms $a$ (‘symmetric algebra’).
   
   All triples are consistent except Peircean transforms of:
   $(1', a, a')$ for $a \neq a'$, and $(r, b, g)$.

These are both relation algebras.
Can you tell if they are in **RRA** or not?
Relation algebra atom structures, complex algebras

Abstractly, a *relation algebra atom structure* is a structure $S = (S, \text{Id}, \circlearrowright, C)$ satisfying certain first-order conditions (the ‘correspondents’ of Tarski’s RA axioms).

If $\mathcal{A}$ is an atomic relation algebra then $\text{At} \mathcal{A}$ is such a structure.

Conversely, given such an $S$, we can form its *complex algebra*:

$$S^+ = (\mathcal{P}(S), \cup, \setminus, \emptyset, S, 1', \circlearrowright, ;),$$

where

- $1' = \{x \in S : S \models \text{Id}(x)\}$
- $\bar{a} = \{\bar{x} : x \in a\}$ (for $a \subseteq S$)
- $a; b = \{z \in S : \exists x \in a \exists y \in b(S \models C(x, y, z))\}$ (for $a, b \subseteq S$).

$S^+$ is an atomic relation algebra, and $\text{At} S^+ \cong S$.

If $\mathcal{A}$ is an atomic relation algebra, $\mathcal{A} \hookrightarrow (\text{At} \mathcal{A})^+$. 
2.2 Duality for representations, by ultrafilters

Atomic RAs are determined by atoms. Are their representations?

Fix any (not necessarily atomic) relation algebra \( \mathcal{A} \), and a representation \( h : \mathcal{A} \to \prod_{i \in I} \mathcal{R}(X_i) \), with base set \( X = \bigcup X_i \).

Each \( a \in \mathcal{A} \) induces a binary relation \( h(a) \) on \( X \). For \( x, y \in X \) let

\[
h_*(x, y) = \{ a \in \mathcal{A} : (x, y) \in h(a) \}.
\]

If \( (x, y) \notin h(1) \) then \( h_*(x, y) = \emptyset \). Otherwise, writing \( f = h_*(x, y) \):

1. \( 1 \in f \)
2. if \( a \in f \) and \( a \leq b \) then \( b \in f \)
3. if \( a, b \in f \) then \( a \cdot b \in f \)
4. for all \( a \in \mathcal{A} \), either \( a \in f \) or \( -a \in f \) (not both).

There is a name for such a subset of a boolean algebra: an *ultrafilter*. So for all \( x, y \in X \), \( h_*(x, y) \) is either \( \emptyset \) or an ultrafilter on \( \mathcal{A} \).
Coherence conditions

The relational operations yield more properties of the $h_*(x, y)$:

5. For any $x, y \in X$, we have $1' \in h_*(x, y)$ iff $x = y$.

6. For any $x, y \in X$, we have $h_*(x, y) = \{a : a \in h_*(y, x)\}$.

7. For any $x, y, z \in X$ and $a \in h_*(x, z)$, $b \in h_*(z, y)$, $c \in h_*(x, y)$, we have $(a; b) \cdot c \neq 0$.

8. For any $x, y \in X$ and $a, b$ in $\mathcal{A}$, if $a; b \in h_*(x, y)$, then there is some $z \in X$ with $a \in h_*(x, z)$ and $b \in h_*(z, y)$.

9. For any non-zero $a \in \mathcal{A}$, there are $x, y \in X$ with $a \in h_*(x, y)$.

These conditions are equivalent to $h$ being a representation of $\mathcal{A}$. This gives us a dual view of representations.
Principal ultrafilters

An ultrafilter \( f \) of a relation algebra \( A \) is \textit{principal} if it is of the form
\[ \{ a \in A : a \geq x \} \]
for some \( x \in A \).

In that case, \( x = \prod f \), and \( x \) is an \textit{atom} of \( A \).

An ultrafilter is principal iff it contains an atom.

- Every ultrafilter of a \textit{finite} relation algebra is principal.
- Every infinite relation algebra has non-principal ultrafilters.
2.3 Bringing them together: representations of finite relation algebras

Let $\mathcal{A}$ be a finite (non-trivial simple) relation algebra. Fix a representation $h: \mathcal{A} \rightarrow \mathcal{R}(X)$ of $\mathcal{A}$. As each $h_*(x, y)$ is principal, let the atom $\prod h_*(x, y)$ stand for it.

So $h$ can be viewed dually very simply, as a complete labelled digraph $M = (X, \lambda)$, where $X$ is a set and $\lambda: X \times X \rightarrow \text{At } \mathcal{A}$ is the labelling. For all $x, y, z \in X$,

- $\lambda(x, y) \leq 1$ iff $x = y$.
- $\lambda(x, y) = \lambda(y, x)^\sim$.
- $\lambda(x, y) \leq \lambda(x, z) ; \lambda(z, y)$. That is, ‘all triangles are consistent’.
- For all $a, b \in \text{At } \mathcal{A}$, if $\lambda(x, y) \leq a ; b$ then there is $w \in X$ with $\lambda(x, w) = a$ and $\lambda(w, y) = b$.
  ‘All consistent triples are witnessed wherever possible.’
2.4 Complete representations

Now let $h$ be a representation of an infinite relation algebra $\mathcal{A}$. The ultrafilters $h_*(x, y)$ need not be principal — even if $\mathcal{A}$ is atomic.

Let’s examine the case when they are all principal.

- $h$ is said to be an **atomic representation** if every ultrafilter $h_*(x, y)$ is principal.
- $h$ is said to be a **complete representation** if it preserves all existing sums: for every $S \subseteq \mathcal{A}$ such that $\sum S$ exists, $h(\sum S) = \bigcup \{h(a) : a \in S\}$.

Fact (exercise):
A representation of a relation algebra is atomic iff it is complete.
Properties of complete representations

1. All representations of \textit{finite} relation algebras are complete.
2. All infinite $\mathcal{A} \in \mathbf{RRA}$ have \textit{incomplete} representations.
3. Any \textit{completely representable} relation algebra (one with a complete representation) is atomic. (Converse fails.)

We can dually characterise any complete representation by a digraph with atoms labeling edges (as for finite relation algebras).

Let $\mathbf{CRA}$ be the class of completely representable relation algebras.

- $\mathbf{CRA}$ is properly contained in $\mathbf{RRA}$.
- $\mathbf{RRA}$ is closure of $\mathbf{CRA}$ under subalgebras (see next slide).
- $\mathbf{CRA}$ is non-elementary (see Robin’s 2nd talk).

\textit{Problem:} does $\mathbf{CRA}$ have the same first-order theory as the class of finite representable relation algebras?
2.5 Canonical extensions (Jónsson–Tarski, 1951)

Let $\mathcal{A}$ be a relation algebra. The set of ultrafilters of $\mathcal{A}$ can be made into a relation algebra atom structure $\mathcal{A}_+$ (the canonical frame of $\mathcal{A}$):

- $f \leq 1$ iff $1^\mathcal{A} \in f$
- $\tilde{f} = \{\tilde{a} : a \in f\}$
- $f ; g \geq h$ iff $a \in f$, $b \in g \Rightarrow a ; b \in h$.

The relation algebra $\mathcal{A}^\sigma := (\mathcal{A}_+)^+$ (complex algebra over the ultrafilters) is called the canonical extension of $\mathcal{A}$.

$\mathcal{A}$ embeds into $\mathcal{A}^\sigma$ via $a \mapsto \{f \in \mathcal{A}_+ : a \in f\}$, so we regard $\mathcal{A} \subseteq \mathcal{A}^\sigma$.

**Fact** (Monk, ~1966): $\mathcal{A} \in \text{RRA} \iff \mathcal{A}^\sigma \in \text{RRA}$. So **RRA** is what’s called a canonical variety.

Indeed, if $\mathcal{A} \in \text{RRA}$ then $\mathcal{A}^\sigma$ has a complete representation.
2.6 (McNeille) completions (Monk 1970)

A **completion** of a relation algebra $\mathcal{A}$ is a relation algebra $\overline{\mathcal{A}}$ such that

1. $\mathcal{A} \subseteq \overline{\mathcal{A}}$,

2. $\overline{\mathcal{A}}$ is complete as a boolean algebra: $\sum S$ exists for all $S \subseteq \overline{\mathcal{A}}$,

3. $\mathcal{A}$ is dense in $\overline{\mathcal{A}}$: $\forall c \in \overline{\mathcal{A}} \setminus \{0\} \ \exists a \in \mathcal{A} \setminus \{0\} \ (a \leq c)$.

Monk (1970) showed that completions of relation algebras always exist and are unique up to isomorphism.

*Easy fact:* If $\mathcal{A}$ is an atomic relation algebra, then its completion is $(\text{At } \mathcal{A})^+$ — the complex algebra over its atom structure.
Some facts about completions

The completion of a relation algebra $\mathcal{A}$ is somewhat analogous to its canonical extension. They both extend $\mathcal{A}$ and are complete as boolean algebras.

However,

- $\mathcal{A}^\sigma$ is always atomic.
  - $\overline{\mathcal{A}}$ is atomic iff $\mathcal{A}$ is atomic.

- $\overline{\mathcal{A}}$ preserves all existing sums and products of elements of $\mathcal{A}$.
  - For infinite $\mathcal{A}$, $\mathcal{A}^\sigma$ never does (consider a non-principal ultrafilter).

- **RRA** is closed under canonical extensions (Monk).
  - We show later that **RRA is not closed under completions**.

So canonical extensions of relation algebras seem the more useful.
Conclusion

We have seen

1. boolean algebras, relation algebras (RA), representable relation algebras (RRA)

2. simple relation algebras: have representations (if any) of form
   \[ h : A \rightarrow \mathcal{R}e(X) \]

3. atomic relation algebras, atom structures, complex algebras

4. representations as graphs with edges labeled by ultrafilters (or atoms, for finite relation algebras)

5. complete representations — when the ultrafilters are principal

6. canonical extensions, completions
Coming next

• Games

• Monk algebras, completions, canonicity

• rainbow algebras: RRA not finitely axiomatisable, CRA non-elementary
Some references


