

Substructural Logics

Introduction

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Nonclassical logics

Broadly speaking, two main directions in the study of nonclassical logics are:

- Logics with additional operators
modal logics, temporal logics, epistemic logics etc.
- Logics with various nonclassical implications

Outline of my talk

- First, some of basic substructural logics are introduced in a **sequent formulation**.
- A substantial part of logics with nonclassical implication is shown to be subsumed into substructural logics.
- Close connections between logic and algebra are briefly discussed by using notions of algebraic logic.

N. Galatos, P. Jipsen, T. Kowalski, HO: [Residuated Lattices: an algebraic glimpse at substructural logics](#), Studies in Logic, vol.151, Elsevier, April, 2007.

Sequent systems

A **sequent** is an expression of the following form with $m \geq 0$.

$$\alpha_1, \dots, \alpha_m \Rightarrow \beta$$

Intuitively, it means " β *follows from assumptions* $\alpha_1, \dots, \alpha_m$ ".

Each sequent system consists of initial sequents (**axioms**) and rules that determine *correct* sequents in the system.

Sequent system LJ

The sequent system **LJ** for intuitionistic logic by Gentzen consists of initial sequents, i.e. sequents of the form $\alpha \Rightarrow \alpha$, and the following three kinds of rules.

- Rules for logical connectives
- Cut
- Structural rules

Algebraic interpretation

An algebraic interpretation of sequents in **LJ** is given by using **Heyting algebras**, so as to satisfy:

A sequent $\alpha_1, \dots, \alpha_m \Rightarrow \beta$ is provable in **LJ** iff

$\mathbf{A} \models \alpha_1 \wedge \dots \wedge \alpha_m \leq \beta$ for every **Heyting algebra** \mathbf{A} .

Rules for lattice operations

Capital Greek letters denote finite sequences of formulas.

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \varphi \quad \Gamma, \beta, \Delta \Rightarrow \varphi}{\Gamma, \alpha \vee \beta, \Delta \Rightarrow \varphi} (\vee \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta} (\Rightarrow \vee 1)$$

$$\frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta} (\Rightarrow \vee 2)$$

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \varphi}{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \varphi} (\wedge 1 \Rightarrow)$$

$$\frac{\Gamma, \beta, \Delta \Rightarrow \varphi}{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \varphi} (\wedge 2 \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta} (\Rightarrow \wedge)$$

Rules for implication and Cut

- Rules for implication

$$\frac{\Gamma \Rightarrow \alpha \quad \beta, \Delta \Rightarrow \varphi}{\Gamma, \alpha \rightarrow \beta, \Delta \Rightarrow \varphi} \quad (\rightarrow \Rightarrow)$$

$$\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta} \quad (\Rightarrow \rightarrow)$$

- Cut

$$\frac{\Gamma \Rightarrow \alpha \quad \Sigma, \alpha, \Xi \Rightarrow \varphi}{\Sigma, \Gamma, \Xi \Rightarrow \varphi} \quad (\text{cut})$$

In algebraic terms, $x \leq a$ and $y \wedge a \wedge z \leq d$ imply $y \wedge x \wedge z \leq d$.

Structural rules

Structural rules determine roles of **commas** in sequents.

(e) exchange rule (**commutativity**):
$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \varphi}{\Gamma, \beta, \alpha, \Delta \Rightarrow \varphi}$$

(c) contraction rule (**square-increasing**):
$$\frac{\Gamma, \alpha, \alpha, \Delta \Rightarrow \varphi}{\Gamma, \alpha, \Delta \Rightarrow \varphi}$$

(i) left weakening rule (**integrality**):
$$\frac{\Gamma, \Delta \Rightarrow \varphi}{\Gamma, \alpha, \Delta \Rightarrow \varphi}$$

(o) right weakening rule (**minimality of 0**):
$$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \alpha}$$

(i) together with (o) is called (w) (weakening rules).

Commas and structural rules

a) Exchange rule allows us to use assumptions in an **arbitrary order**:

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \varphi}{\Gamma, \beta, \alpha, \Delta \Rightarrow \varphi}$$

b) Without contraction rule, every (occurrence of each) assumption is used **at most once** in deriving a conclusion:

$$\frac{\Gamma, \alpha, \alpha, \Delta \Rightarrow \varphi}{\Gamma, \alpha, \Delta \Rightarrow \varphi}$$

c) Without weakening rule (i), every assumption is used **at least once** in deriving a conclusion:

$$\frac{\Gamma, \Delta \Rightarrow \varphi}{\Gamma, \alpha, \Delta \Rightarrow \varphi}$$

Commas in LJ

By the help of contraction and (left) weakening of **LJ**, it is shown that a sequent

$$\alpha_1, \dots, \alpha_m \Rightarrow \beta$$

is provable in **LJ** iff

$$\alpha_1 \wedge \dots \wedge \alpha_m \Rightarrow \beta$$

is provable in **LJ**. Thus, commas of **LJ** can be understood as **conjunctions**.

This gives a justification of our algebraic interpretation of sequents in Heyting algebras.

Full Lambek calculus FL

- We get a sequent system for classical logic by adding sequents of the form $\neg\neg\alpha \Rightarrow \alpha$ as initial sequents to LJ.
- Our basic sequent system FL (Full Lambek calculus) is obtained from LJ by deleting all of structural rules.

Comma and fusion

Commas in **FL** cannot be expressed by conjunctions anymore.

We introduce a new logical connective, called **fusion** (\cdot , in symbol) to represent commas explicitly. Rules for fusion are:

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha \cdot \beta} (\Rightarrow \cdot) \quad \frac{\Sigma, \alpha, \beta, \Gamma \Rightarrow \varphi}{\Sigma, \alpha \cdot \beta, \Gamma \Rightarrow \varphi} (\cdot \Rightarrow)$$

Then, in **FL**:

$\alpha \cdot \beta \Rightarrow \varphi$ is provable iff $\alpha, \beta \Rightarrow \varphi$ is provable.

Implications as residuals

Without exchange, the implication will be splitted into two, **left-** and **right-residual** ($\alpha \backslash \beta$ and β / α , in symbol).

$$\frac{\Gamma \Rightarrow \alpha \quad \Xi, \beta, \Delta \Rightarrow \varphi}{\Xi, \Gamma, \alpha \backslash \beta, \Delta \Rightarrow \varphi} (\backslash \Rightarrow) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \backslash \beta} (\Rightarrow \backslash)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Xi, \beta, \Delta \Rightarrow \varphi}{\Xi, \beta / \alpha, \Gamma, \Delta \Rightarrow \varphi} (/ \Rightarrow) \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \beta / \alpha} (\Rightarrow /)$$

Then, the following **residuation** relation holds in **FL**.

$\alpha, \beta \Rightarrow \varphi$ is provable iff $\beta \Rightarrow \alpha \backslash \varphi$ is provable iff
 $\alpha \Rightarrow \varphi / \beta$ is provable.

Substructural logics

Substructural logics (over **FL**) are axiomatic extensions of **FL**. (Each sequent system is often identified with the set of formulas provable in it.)

Basic substructural logics can be defined by adding some of structural rules to **FL**.

- **FL** — deleting all structural rules from **LJ**
- **FL_e** — **FL** + exchange ($\mathbf{FL} + \alpha \cdot \beta \Rightarrow \beta \cdot \alpha$)
- **FL_c** — **FL** + contraction ($\mathbf{FL} + \alpha \Rightarrow \alpha \cdot \alpha$)
- **FL_{ew}** — **FL** + exchange + weakenings
- **MALL** — **FL_e** + $\neg\neg\alpha \rightarrow \alpha$

Important substructural logics

- Lambek calculus — logic without structural rules

Calculus for categorial grammar introduced by Ajdukiewicz and Bar-Hillel (J.Lambek, 1958), rediscovered in 80s (J. van Benthem and W. Buszkowski).

- Relevant logics — logics without weakening rules

A. Anderson, N. Belnap Jr., R.K. Meyer, M. Dunn, A. Urquhart etc.

- Logics without contraction rule

V. Grishin (middle of 1970), H.O. - Y. Komori (1985), Łukasiewicz's many-valued logics, fuzzy logics

- Linear logic — logic only with exchange rule

J.-Y. Girard (1987)

- Johansson's minimal logic — logic without right-weakening

Algebras for FL

Similarly to **LJ**, the algebraic interpretation of a sequent of **FL** must satisfy the following:

A sequent $\alpha_1, \dots, \alpha_m \Rightarrow \beta$ is provable in **FL** iff

$\mathbf{A} \models \alpha_1 \cdot \dots \cdot \alpha_m \leq \beta$ for every algebra **A**.

What algebras are suitable for **FL**? Apparently, they must be **partially ordered monoids**.

Residuated structures

A p.o. monoid is a structure $\langle L; \cdot, 1; \leq \rangle$ such that

- $\langle L; \leq \rangle$ is a p.o. set,
- $\langle L; \cdot, 1 \rangle$ is a monoid such that

$$x \leq y \Rightarrow xz \leq yz \text{ and } zx \leq zy.$$

A p.o. monoid is **residuated** if there exist division operations \backslash and $/$ such that

$$xy \leq z \Leftrightarrow x \leq z/y \Leftrightarrow y \leq x \backslash z$$

Residuated lattices

An algebra $\langle L; \wedge, \vee, \cdot, \backslash, /, 1 \rangle$ is a **residuated lattice** if

- $\langle L; \cdot, \backslash, /; \leq \rangle$ is a residuated p.o. monoid,
- $\langle L; \wedge, \vee \rangle$ forms a lattice.

Residuated lattices are **equationally definable**. In particular, the law of residuation is expressed by equations;

$$x(x \backslash z \wedge y) \leq z \text{ and } y \leq x \backslash (xy \vee z), \text{ etc.}$$

An **FL-algebra** is a residuated lattice with a fixed element **0**. Two **negations** are introduced by $\sim x = x \backslash 0$ and $-x = 0/x$. The variety of **FL**-algebras is denoted by \mathcal{FL} .

Why sequent systems for SLs?

In sequent formulation, a monoid operation is introduced explicitly as **comma**, and moreover, **implication(s)** behaves exactly as its residual(s).

- Differences of **commas**, typically expressed by **structural rules**, cause differences of **implications**, and vice versa.
- Any transparent Hilbert-style system or natural deduction for the logic **FL** is not known (probably impossible).

Implications and **monoids** behave exactly in the same way as **divisions** and **multiplications** in arithmetic.

Cut elimination

Cut elimination is the most important property of a sequent system. **Cut elimination** for a sequent system **S** means:

- *if a sequent is provable in **S** then it is also provable in **S** without using cut rule.*

Cut elimination holds for most of basic substructural logics introduced so far, though a very limited number of substructural logics has a **cut-free** sequent system.

J. Lambek (1958), S. Tamura (1974), H.O-Y. Komori (1985) etc.

Consequences of cut elimination

- Craig's interpolation property (Maehara's method)
- Disjunction property (of logics without right contraction rule)
- Decidability
- **Subformula property** — If a sequent $\Gamma \Rightarrow \alpha$ is provable, it has such a proof that every formula in the proof is a subformula of a formula in $\Gamma \Rightarrow \alpha$.

Decidability I

For a given sequent $\Gamma \Rightarrow \alpha$, we will search for its cut-free proof by the **decompositions**, i.e. applications of rules of **FL** in the reverse direction.

- Sometimes **backtracking** process becomes necessary in searching proofs, as there are several choices in applying a rule.
- Since **decomposed sequents are always simpler** than the original one, every decomposition will eventually terminate.
- When no further decompositions are applicable, check whether each sequent at the top is an initial one or not.

Decidability II

This **proof-search algorithm** terminates eventually, since there are only finite number of possible decompositions.

- If all of them are initial sequents then this gives us a required cut-free proof of $\Gamma \Rightarrow \alpha$.
- On the other hand, if every such **trial** fails, then the sequent is not provable.

Thus

***FL** is decidable.*

Similar arguments work also for **FL_e** and **FL_{ew}**.

Decidability III

On the other hand, the presence of **contraction rule** causes some difficulties in searching proofs, since upper sequents are **not always simpler** than the lower one.

- For **LJ**, these difficulties are avoided by considering only **reduced sequents** (by G. Gentzen).
- For **FL_{ec}** we need **Curry's lemma** and **Kripke's lemma** to overcome difficulties (by S. Kripke). In fact, this decision procedure is of high computational complexity.
- Moreover, contraction rule with **distributive law** causes even the undecidability (by A. Urquhart).

Deducibility in FL

For a set of formulas Σ and a formula α , α is **deducible** from Σ in **FL** ($\Sigma \vdash_{\mathbf{FL}} \alpha$) iff:

the sequent $\Rightarrow \alpha$ is provable in **FL** when adding sequents $\Rightarrow \gamma$ (for each $\gamma \in \Sigma$) as extra initial sequents.

Deducibility is different from Provability. For example, while $\alpha \Rightarrow \alpha^2$ is not provable in **FL**, $\alpha \vdash_{\mathbf{FL}} \alpha^2$ holds as;

$$\frac{\Rightarrow \alpha \quad \Rightarrow \alpha}{\Rightarrow \alpha \cdot \alpha} (\Rightarrow \cdot)$$

Deducibility and provability

Can the deducibility relation be reduced to the provability?

Yes, for both classical and intuitionistic logics. In fact, the following **deduction theorem (DT)** holds for them:

$$\Sigma, \alpha \vdash \beta \text{ iff } \Sigma \vdash (\alpha \rightarrow \beta).$$

By this, the decidability of the deducibility follows from that of the provability.

Parameterized local DT

This is not always the case. Still, the following **parameterized local deduction theorem (PLDT)** holds for **FL**. (cf. Czelakowski-Dziobiak)

$\Sigma, \alpha \vdash_{\mathbf{FL}} \beta$ iff there exist **iterated conjugates** δ_i of α ($i \leq m$ for some m) such that $\Sigma \vdash_{\mathbf{FL}} (\prod \delta_i) \setminus \beta$.

Here, each iterated conjugate of α is obtained from α by applying the left-conjugate $\lambda_\theta(\alpha) = (\theta \setminus \alpha \theta) \wedge 1$ and/or the right-conjugate $\rho_\phi(\alpha) = (\phi \alpha / \phi) \wedge 1$ with some parameters θ, ϕ, \dots , repeatedly.

Parameterized local DT

The proof goes essentially in the same way as DT for **LJ**. Though **FL** has no structural rules, we can **simulate** both weakening and exchange rules (but not contraction).

- if $\Gamma, \Delta \Rightarrow \theta$ is provable then $\Gamma, \psi \wedge 1, \Delta \Rightarrow \theta$ is provable,
- if $\Gamma, \alpha, \beta, \Delta \Rightarrow \theta$ is provable then both $\Gamma, \beta, \lambda_\beta(\alpha), \Delta \Rightarrow \theta$ and $\Gamma, \rho_\alpha(\beta), \alpha, \Delta \Rightarrow \theta$ are provable.

$$b(\lambda_b(a)) = b((b \setminus ab) \wedge 1) \leq b(b \setminus ab) \leq ab$$

Local deduction theorem

In a system with **exchange rule**, **conjugates** are not necessary. Thus, PLDT can be simplified into the following **local deduction theorem**.

$$\Sigma, \alpha \vdash_{\mathbf{FL}_e} \beta \text{ iff } \Sigma \vdash_{\mathbf{FL}_e} (\alpha \wedge 1)^m \rightarrow \beta \text{ for some } m.$$

It is still **local**, as we cannot always determine such an m from given Σ, α, β .

Decision problem of deducibility

The deducibility relation for a logic \mathbf{L} is *decidable* iff

*there is an effective procedure of deciding whether or not $\Sigma \vdash_{\mathbf{L}} \alpha$ holds for each **finite** set of formulas Σ and each formula α .*

Thus, the decision problem of the deducibility relation for a logic \mathbf{L} is equivalent to the decision problem of **quasi-equational theory** of the corresponding variety $V(\mathbf{L})$.

Undecidability

The following follows essentially from the proof of the undecidability of linear logic (with **exponentials**) by Lincoln, Mitchell, Scedrov & Shankar.

The deducibility relation for FL_e is undecidable.

Buszkowski showed that this holds even for the $\{\rightarrow, \cdot, \wedge\}$ fragment (by personal communication).

Substructural logics

We give an alternative definition of a **substructural logic over FL** (as a set of formulas).

A set of formula Σ is **deductively closed** w.r. to $\vdash_{\mathbf{FL}}$, if $\Pi \vdash_{\mathbf{FL}} \beta$ for a subset Π of Σ then $\beta \in \Sigma$.

A set of formulas \mathbf{L} is a **substructural logic** iff

- it is deductively closed w.r. to $\vdash_{\mathbf{FL}}$,
- it is closed under substitution.

More on Logic and Algebra

The equational consequence $\{u_i = v_i; i \in I\} \models_{\mathcal{V}} s = t$ of a subvariety \mathcal{V} of \mathcal{FL} is defined for a set of equations $\{u_i = v_i; i \in I\} \cup \{s = t\}$ by

for each algebra \mathbf{A} in \mathcal{V} and each assignment f on A , $f(s) = f(t)$ holds whenever $f(u_i) = f(v_i)$ holds for all $i \in I$.

In particular, $\{u_i = v_i; 1 \leq i \leq m\} \models_{\mathcal{V}} s = t$ is equivalent to the validity of the following **quasi-equation** in every \mathbf{A} in \mathcal{V} .

$(u_1 = v_1 \text{ and } \dots \text{ and } u_m = v_m) \text{ implies } s = t.$

Algebraization theorem

The deducibility relation corresponds exactly to the equational consequence.

1. For each subvariety \mathcal{V} of \mathcal{FL} , $\{u_i = v_i; i \in I\} \models_{\mathcal{V}} s = t$ iff $\{u_i \setminus v_i \wedge v_i \setminus u_i; i \in I\} \vdash_{L(\mathcal{V})} s \setminus t \wedge t \setminus s$,
2. Conversely, for each substructural logic \mathbf{L} , $\{\beta_j; j \in J\} \vdash_{\mathbf{L}} \alpha$ iff $\{1 \leq \beta_j; j \in J\} \models_{V(\mathbf{L})} 1 \leq \alpha$,
3. Moreover, they are mutually inverse transformations.

In abstract algebraic logic, we say this as:

for each substructural logic \mathbf{L} , $\vdash_{\mathbf{L}}$ is algebraizable and $\mathcal{V}(\mathbf{L})$ is an equivalent algebraic semantics for it.

Appendices

Constructive reasoning

Mathematical arguments are often infinitary and non-constructive. From intuitionists' viewpoint:

mathematical arguments must be constructive.

- To infer $\alpha \rightarrow \beta$, there must be an **algorithm** for constructing a proof of β from any given proof of α ,
- To infer $\alpha \vee \beta$, it is necessary to say which of α and β holds, and also to show **evidences**.

Because of them, both the law of double negation $\neg\neg\alpha \rightarrow \alpha$ and the law of excluded middle $\alpha \vee \neg\alpha$ are rejected.

Relevant reasoning

The implication in classical logic is **material implication**, i.e. $\alpha \rightarrow \beta$ is identified with $\neg\alpha \vee \beta$. Thus, both

- $(\alpha \wedge \neg\alpha) \rightarrow \beta$ (*a falsehood implies every proposition*)
- $\beta \rightarrow (\alpha \rightarrow \alpha)$ (*a truth is implied by every proposition*)

are classically valid. But their validity will be counterintuitive.

Relevant logics are formal systems of **relevant implication**, i.e. "implication" used in our daily reasoning.

Many-valued logics

In 1920s, J. Łukasiewicz introduced both $n + 1$ -valued logic (for each $n > 0$) with the set of truth values $\{0, 1/n, 2/n, \dots, (n - 1)/n, 1\}$, and also infinite-valued logic with the unit interval $[0, 1]$ as the set of truth values.

The truth table of each connective is defined as follows:

$$a \wedge b = \min\{a, b\}$$

$$\neg a = 1 - a$$

$$a \vee b = \max\{a, b\}$$

$$a \rightarrow b = \min\{1, 1 - a + b\}$$

$$= \begin{cases} 1 & a \leq b \\ 1 - a + b & a > b \end{cases}$$

Fuzzy logics

Fuzzy logics are introduced by using **triangular norms** (t-norms). A **t-norm** is a binary operation on $[0, 1]$ which is **associative, commutative, monotone** and has the unit 1.

For each given t-norm \cdot , a **fuzzy implication** \rightarrow is introduced by

$$a \rightarrow b = \sup\{z : a \cdot z \leq b\}$$

whenever it is **left-continuous**. Obviously, the following holds.

$$a \cdot b \leq c \text{ iff } b \leq a \rightarrow c$$