Representing Relation Algebras, by Games

Hirsch and Hodkinson, London

January 6, 2009
Binary relations $\simeq >$ *(algebraization)* $\mathcal{RA}$

Is $\mathcal{A}$ representable? — very hard.
Types of representations

- Classical representations.
- Complete representations.
- Relativized representations.
- Representations of a reduced signature.
- Homogeneous representations.
- Finite representations.
**Characterising representability**

One approach: find first-order theory (or better, an equational theory) $\Delta$ such that

$$\mathcal{A} \models \Delta \iff \mathcal{A} \text{ has approp. rep.}$$

This may or may not be possible, and it is almost always fearsomely difficult.
Characterising representability by games

Our approach: devize two player game $G$ such that

$\exists$ has a w.s. in $G(A) \iff A$ has an approp. rep.

Actually, in many cases we can use these games to obtain first-order theories as above.
Abelard and Héloïse
Outline of the talk

1. Games to test representability — finite algebra case.

2. Obtaining a first-order theory from the game.

3. Atomic relation algebras (countably many atoms).
   Game for *complete representability*

4. Arbitrary classical representation game.

5. Other games.
Representation

\[ A = (A, 0, 1, +, -, 1', \sim, ;) \]

\[ h : A \rightarrow \wp(X \times X) \]

such that

\[ a \neq 0 \Rightarrow h(a) \neq \emptyset \quad (h \text{ is } 1-1) \]

\[ h(0) = \emptyset \]

\[ h(a + b) = h(a) \cup h(b) \]

\[ h(-a) = h(1) \setminus h(a) \]

\[ h(1') = \{ (x, x) : x \in X \} \]

\[ (x, y) \in h(a\sim) \iff (y, x) \in h(a) \]

\[ (x, y) \in h(a; b) \iff \exists z \ [(x, z) \in h(a) \land (z, y) \in h(b)] \]
Square representations of simple algebras

If

\[ h(1) = X \times X \]

then \( h \) is a square representation.

If \( \mathcal{A} \) is simple and representable then it has a square representation.
Every relation algebra is a subalgebra of a product of simple relation algebras.
(x, y) ∈ h(1) ⇒ ∃α ∈ At(\mathcal{A}) (x, y) ∈ h(\alpha)

If \( h \) is a square representation of \( \mathcal{A} \), can define a labelled graph \((X, \lambda)\) by

\[
\lambda : X \times X \to At(\mathcal{A}) \\
\lambda(x, y) = \prod_{a \in \mathcal{A}} \{a \in \mathcal{A} : (x, y) \in h(a)\}
\]

Conversely, if \( \lambda : X \times X \to At(\mathcal{A}) \) satisfies

\[
\lambda(x, y) \leq 1' \iff x = y \\
\lambda(x, y) = \lambda(y, x) \\
\lambda(x, z) ; \lambda(z, y) \geq \lambda(x, y)
\]

and for all atoms \( \alpha, \beta \in At(\mathcal{A}) \),

\[
\lambda(x, y) \leq \alpha ; \beta \Rightarrow \exists z [\lambda(x, z) = \alpha \land \lambda(z, y) = \beta]
\]

then \( \lambda \) defines a square representation \( h \), by

\[
h(a) = \{(x, y) : a \geq \lambda(x, y)\}
\]
Atomic $\mathcal{A}$-network: $N = (X, \lambda)$

\[ \lambda : X \times X \to At(\mathcal{A}) \]

satisfies

\[ \lambda(x, y) \leq 1' \iff x = y \]
\[ \lambda(x, y)\sim = \lambda(y, x) \]
\[ \lambda(x, z); \lambda(z, y) \geq \lambda(x, y) \]

But maybe there are nodes $x, y$ and atoms $a, b$ such that

\[ \lambda(x, y) \leq a; b \text{ yet } \exists z [\lambda(x, z) = a \land \lambda(z, y) = b] \]

Then $(x, y, a, b)$ is a defect of the atomic network.

Write $N$ instead of $X$ or $\lambda$. 
Games on atomic $\mathcal{A}$-networks

Two players: $\forall$ and $\exists$. The game $G_n^a(\mathcal{A})$ has $n$ rounds (where $n \leq \omega$). A play of the game will be

$$N_0 \subseteq N_1 \subseteq \ldots \subseteq N_{t-1} \subseteq N_t \subseteq \ldots \quad (t < n)$$

Round 0:

- $\forall$ picks $a_0 \in \text{At} \mathcal{A}$.

- $\exists$ plays an atomic network $N_0$ with $a_0$ occurring as a label in it.

Round $t$ ($1 \leq t < n$): Suppose that the current atomic network at the start of the round is $N_{t-1}$. Play goes as follows:
Round \( t \) of \( G^\alpha_n(A) \)

\( \forall \) picks \( x, y \in N_{t-1} \) and \( a, b \in \text{At}(A) \) with \( a \wedge b \geq N_{t-1}(x, y) \)
Round $t$ of $G^\alpha_n(A)$

\[ \forall \text{ picks } x, y \in N_{t-1} \text{ and } a, b \in \text{At}(A) \text{ with } a; b \geq N_{t-1}(x, y) \]

∃ responds with...
Round $t$ of $G_{n}^{\alpha}(A)$

∀ picks $x, y \in N_{t-1}$ and $a, b \in \text{At}(A)$ with $a ; b \geq N_{t-1}(x, y)$

∃ responds with...

...an atomic network $N_{t}$, extending $N_{t-1}$, & containing some node $z$ such that $N_{t}(x, z) = a, N_{t}(z, y) = b$
Who wins?

In any round, if ∃ cannot play, or if she plays a labelled graph that fails to be an atomic network, then ∀ wins.

If ∃ plays a legitimate atomic network in each round then she wins.
Characterising representability for finite RAs, by games

**Theorem 1** Let \( A \) be a finite relation algebra.

1. \( A \in \text{RRA} \) iff \( \exists \) has a winning strategy in \( G_\omega^a(A) \).

2. \( \exists \) has a winning strategy in \( G_\omega^a(A) \) iff she has one in \( G_n^a(A) \) for all finite \( n \).

3. One can construct first-order sentences \( \sigma_n \) for \( n < \omega \) (independently of \( A \)) such that \( A \Vdash \sigma_n \) iff \( \exists \) has a winning strategy in \( G_n^a(A) \).

Conclude that for a finite relation algebra \( A \),

\[
A \in \text{RRA} \iff A \Vdash \{ \sigma_n : n < \omega \}.
\]
The axioms $\sigma_n$ (sketch)

Given an atomic network $N$, and $k < \omega$, we write an axiom $\tau_k(N)$ saying that $\exists$ can win $G^\alpha_k(A)$ starting from $N$. We go by induction on $k$. All quantifiers are implicitly relativised to atoms.

$$
\tau_0(N) = \bigwedge_{x \in N} (N(x, x) \leq 1') \\
\land \bigwedge_{y \in N \setminus \{x\}} N(x, y) \not\leq 1') \\
\land \bigwedge_{x, y \in N} N(x, y) = N(y, x)^\sim \\
\land \bigwedge_{x, y, z \in N} N(x, y) \leq N(x, z); N(z, y).
$$

$$
\tau_{k+1}(N) = \bigwedge_{x, y \in N} \forall a, b(N(x, y) \leq a; b \rightarrow \exists N' \supseteq N \\
(\tau_k(N') \land \bigvee_{z \in N'} (N'(x, z) = a \\
\land N'(z, y) = b))).
$$

$$
\sigma_k = \forall a_0 \exists N(\tau_{k-1}(N) \land \bigvee_{x, y \in N} N(x, y) = a_0)
$$
McKenzie’s algebra

4 atoms: $1’, <, >, \#$.

$1’ \equiv 1’, \quad < \equiv >, \quad > \equiv <, \quad \# \equiv \#$

All triples are consistent except Peircean transforms of:
$(1’, a, a’) \text{ for } a \neq a’, (\langle, \langle, \rangle), (\langle, \langle, \#), (\#, \#, \#)$. 


McKenzie’s algebra

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![Diagram of McKenzie’s algebra]
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![Diagram of McKenzie’s algebra]

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McKenzie’s algebra

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23
### McKenzie’s algebra

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∀ wins.
Anti-Monk algebra (∀’s first kind of move)

4 atoms: 1’, r, b, g.

\( x \sim = x \) for all atoms \( x \) (‘symmetric algebra’)

All triples are consistent except Peircean transforms of:
(1’, \( a, a' \)) for \( a \neq a' \), and (r, b, g).
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Anti-Monk algebra (∀’s second kind of move)

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All triples are consistent except Peircean transforms of: 
(1’, a, a’) for \( a \neq a' \), and (r, b, g).
Anti-Monk algebra (\(\forall\)'s second kind of move)

4 atoms: 1’, \(r\), \(b\), \(g\).

\(x \sim x\) for all atoms \(x\) (‘symmetric algebra’)

All triples are consistent except Peircean transforms of:
(1’, \(a\), \(a'\)) for \(a \neq a'\), and \((r, b, g)\).
Anti-Monk algebra (∀’s second kind of move)

4 atoms: 1’, r, b, g.

\[ x \sim = x \] for all atoms \( x \) (‘symmetric algebra’)

All triples are consistent except Peircean transforms of:
(1’, a, a’) for \( a \neq a' \), and \( (r, b, g) \).
Hence

1. McKenzie’s algebra $\mathcal{K} \not\in \text{RRA}$.

   So $\text{RRA} \subset \text{RA}$, as Lyndon (1950) showed.

   In fact, $\mathcal{K}$ is one of the smallest non-representable relation algebras. All relation algebras with $\leq 3$ atoms are representable.

2. The anti-Monk algebra $\mathcal{M} \in \text{RRA}$.

   Exercise: show that if $(X, \lambda)$ is any representation of $\mathcal{M}$, then $X$ is infinite.

   This is perhaps surprising, given that $\mathcal{M}$ is symmetric.
Infinite RAs and Complete representations

If $A$ is infinite and $(x, y) \in h(1)$ then there might not be an atom $\alpha$ with $(x, y) \in h(\alpha)$.
If $(x, y) \in h(1) \Rightarrow \exists \alpha \in \text{At}(A), (x, y) \in h(\alpha)$ then $h$ is called a complete representation of $A$. Equivalently,

$$h(\sum S) = \bigcup_{a \in S} h(a)$$

whenever the supremum $\sum S$ exists in $A$.

If $A$ is atomic with countably many atoms then

$$A \in \text{CRA} \iff \exists \text{ w.s. in } G_{\omega}^{a}(A) \Rightarrow A \models \{\sigma_n : n < \omega\}$$
Another view of a representation

Representation

\[ h : \mathcal{A} \rightarrow \wp(X \times X) \]

defines a labelled graph with nodes \( X \).

\[ \lambda : X \times X \rightarrow \wp(\mathcal{A}) \]

\[ \lambda(x, y) = \{ a : (x, y) \in h(a) \} \]

which satisfies

\[ 0 \not\in \lambda(x, y) \]

\[ a \in \lambda(x, y), \ a \leq b \ \Rightarrow \ b \in \lambda(x, y) \]

\[ a \in \lambda(x, y) \text{ and } b \in \lambda(x, y) \ \Rightarrow \ (a.b) \in \lambda(x, y) \]

\[ 1 \in \lambda(x, y) \ \Rightarrow \ a \in \lambda(x, y) \text{ or } \neg a \in \lambda(x, y) \]

i.e. \( \lambda(x, y) \) is either empty or an ultrafilter, and

\[ 1' \in \lambda(x, y) \iff x = y \]

\[ a^{-} \in \lambda(x, y) \iff a \in \lambda(y, x) \]

\[ a; b \in \lambda(x, y) \iff \exists z \ [a \in \lambda(x, z) \land b \in \lambda(z, y)] \]
Ultrafilter Networks

For a general representation $h$ of $\mathcal{A}$ and $(x, y) \in h(1)$, there must be an ultrafilter $\gamma$ such that

$$a \in \gamma \iff (x, y) \in h(a)$$

Ultrafilter network $N = (X, \lambda)$.

$$\lambda : X \times X \rightarrow uf(\mathcal{A})$$

$$1' \in \lambda(x, y) \iff x = y$$

$$a \sim \in \lambda(x, y) \iff a \in \lambda(y, x)$$

$$a \in \lambda(x, z) \land b \in \lambda(z, y) \Rightarrow a; b \in \lambda(x, y)$$

Defect $(x, y, a, b)$: if $a; b \in \lambda(x, y)$ but $\not\exists z \ (a \in \lambda(x, z) \land b \in \lambda(z, y))$. 

36
Ultrafilter Game $G^u$

Play:

$$N_0 \subset N_1 \subset \ldots \subset N_{t-1} \subset N_t \subset \ldots$$

Round 0:

- $\forall$ picks non-zero $a \in \mathcal{A}$.

- $\exists$ plays an ultrafilter network $N_0$ such that $a \in N_0(0,1)$.

Later, round $t$, current u-network is $N_{t-1}$. $\forall$ picks $x, y \in N_{t-1}$, $a, b \in \mathcal{A}$ such that $a; b \in N_{t-1}(x, y)$. 


Round \( t \) of \( G_u^n(A) \) \((1 \leq t < n)\):

\[ \forall \text{ picks } x, y \in N_{t-1} \text{ and } a, b \in A \text{ with } a ; b \in N_{t-1}(x, y) \]
Round $t$ of $G_n^u(A)$ ($1 \leq t < n$):

$\forall$ picks $x, y \in N_{t-1}$ and $a, b \in A$ with $a \cdot b \in N_{t-1}(x, y)$

$\exists$ responds with...
Round $t$ of $G^u_n(\mathcal{A})$ ($1 \leq t < n$):

$\forall$ picks $x, y \in N_{t-1}$ and $a, b \in \mathcal{A}$ with $a \cdot b \in N_{t-1}(x, y)$

$\exists$ responds with...

$a \in N_t(x, z)$

$b \in N_t(z, y)$

...an atomic network $N_t$, extending $N_{t-1}$, & containing some node $z$ such that $a \in N_t(x, z)$, $b \in N_t(z, y)$
Ultrafilter Game and Representability

Theorem 2  Let $\mathcal{A}$ be an RA. TFAE.

- $\mathcal{A} \in \text{RRA}$

- $\exists$ has winning strategy in $G^\text{ru}_\omega(\mathcal{A})$. 
Representation games

But cannot get first-order axioms directly from this. Need *second order variables* for ultrafilters.
The general representation game

Let $\mathcal{A}$ be arbitrary relation algebra. 

An $\mathcal{A}$-network $N = (X, \lambda)$

$$\lambda : X \times X \to \mathcal{A}$$

satisfies

1. $0 \not\in \lambda(x, y)$
2. $\lambda(x, y) \leq 1' \iff x = y$
3. $\lambda(x, y) \, \vdash = \lambda(y, x)$
4. $\lambda(x, z); \lambda(z, y) \cdot \lambda(x, y) \neq 0$

(all $x, y, z \in X$).
Networks, notation

\[ M = (Y, \mu) \] extends \[ N = (X, \lambda) \] if

- \( Y \supseteq X \) and

- \( M(x, y) \leq N(x, y), \) for \( x, y \in X. \)

Write \( N \) instead of \( X \) or \( \lambda. \) Write \( M \supseteq N \) if \( M \) extends \( N. \)
The general representation game $G$

Play:

$$N_0 \subseteq N_1 \subseteq \ldots \subseteq N_{t-1} \subseteq N_t \subseteq \ldots$$

Round 0:

- $\forall$ picks non-zero $a \in A$.
- $\exists$ plays a network $N_0$ such that $N_0(0, 1) \leq a$. 
∀’s question

Later, round $t$, current network is $N_{t-1}$. ∀ picks $x, y \in N_t$ and $a, b \in A$. ∀ asks:

“Does $a; b \in N(x, y)$?”
Round $t$ of $G_{n}(A)$ ($1 \leq t < n$):

\[ \forall \text{ picks } x, y \in N_{t-1} \text{ and } a, b \in A \]
**Round $t$ of $G_n(A)$ ($1 \leq t < n$):**

\[ \forall \text{ picks } x, y \in N_{t-1} \text{ and } a, b \in A \]

\[ \exists \text{ responds with either } \ldots \]
Round $t$ of $G_n(A)$ ($1 \leq t < n$):

$\forall$ picks $x, y \in N_{t-1}$ and $a, b \in A$

$\exists$ responds with either...

$N_t(x, z) = a$

$N_t(z, y) = b$

...“YES”: a network $N_t$, extending $N_{t-1}$, & containing some node $z$ such that $N_t(x, z) = a$, $N_t(z, y) = b$ and $N_t(x, y) = (a; b).N_{t-1}(x, y)$
Round $t$ of $G_n(A)$ ($1 \leq t < n$):

\[ \forall \text{ picks } x, y \in N_{t-1} \text{ and } a, b \in A \]

\[ \exists \text{ responds with either...or...} \]

\[ \ldots \text{“NO”: a network } N_t \supseteq N_{t-1}, \]
\[ N_t(x,y) = N_{t-1}(x,y) - (a;b) \]
Axiomatising RRA

As before, can write a formula (this time, an equation) $\rho_n$ such that

$$\exists \text{ has w.s. in } G_n(A) \iff A \models \rho_n$$

Theorem 3  The following are equivalent

1. $A \in \text{RRA}$,

2. $\exists \text{ has winning strategy in } G_\omega(A)$,

3. $\exists \text{ has winning strategy in } G_n(A), \text{ all } n < \omega$,

4. $A \models \{ \rho_n : n < \omega \}$. 
Other Games

- Relativized — partial labelling of networks, maybe not transitive.

- $n$-square — $n$-pebble game.

- Reduct — different labels and network consistency properties.

- Homogeneous — amalgamation moves.

- Finite — $\exists$ picks $n < \omega$ initially and must use nodes $\{0, 1, \ldots, n - 1\}$ only thenceforth.
4. Games in algebraic logic: pros and cons

Games provide a simple practical test for representability.
(It is also very useful for theoretical purposes.)

Games can be used to produce axioms as well.

Games on relation algebras generalise to games for other kinds of algebras of relations — e.g., complex algebras (Mikulás, Venema).

In some cases, they can be used to find finite axiomatisations.

Sometimes, a winning strategy can be extracted and used for other things: e.g., decidability, complexity, finite model property.
More pros

Games can suggest some fairly sophisticated constructions of relation algebras. These can be used to prove:

1. \textbf{RRA} is not finitely axiomatisable (Monk, 1964).

2. \textbf{RRA} is not axiomatisable by equations using finitely many variables altogether (Jónsson, 1988)(?).

3. \textbf{RRA} is not closed under completions (Hodkinson, 1997), and hence is not Sahlqvist-axiomatisable (Venema, 1997).

4. In first-order logic, more 3-variable sentences are provable with $n + 1$ variables than with $n$ variables, for all $n \geq 3$. 
5. For a finite relation algebra \( A \), it is undecidable whether \( A \in \text{RRA} \) (Hirsch–Hodkinson, 1999).

6. \text{RRA} is canonical (Monk), but any first-order axiomatisation of it has infinitely many non-canonical axioms (Hodkinson–Venema, 2002).
We use games as a *construction method*, essentially forcing, to build representations of relation algebras.

In general, the representation so obtained is *infinite*. These games are not good at building *finite representations*.

Example: suppose $\mathcal{A}$ is a finite relation algebra with a *flexible atom*, $f$, say. $(a, b, f)$ is consistent for all atoms $a, b \neq 1$.

The game shows $\mathcal{A}$ is representable: $\exists$ can win by using $f$.

**Open problem (Maddux):** must $\mathcal{A}$ have a *finite* representation?

**General issue:** Find ways of constructing finite representations. Can we combine games with, e.g., probabilistic constructions?
Some algebraic logicians dislike games. They prefer the traditional ‘step-by-step’ way, enumerating the requirements of a construction and dealing with them one by one.

But Wilfrid Hodges comments: ‘The notion of a game has to do with people acting together, setting themselves and each other tasks. As a result, game-theoretic versions of mathematical ideas often have a direct intuitive appeal when compared with more formalistic treatments. In the period 1900–1950 logic was fighting to establish itself as a serious branch of mathematics, and if you want your mathematics to be serious you don’t start by talking about people setting up competitions or exercise sessions. Today logic has won its battle for recognition, and [we] can afford to make intuitiveness one of [our] chief aims.’
5. Conclusion

Games provide a simple intrinsic characterisation of the representable relation algebras.

The idea has many precursors — in particular, Lyndon (1950).

Games can be used with many other kinds of algebras of relations.

Games can guide quite sophisticated constructions.

And so they have given great insight into what makes relation algebras tick.
Some references


