Strategic reasoning in compositional games

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First order truth checking

Given:

- First order formula α,
- First order structure $\mathfrak{A} = (S, \tau)$,

Question: Does $\mathfrak{A} \models \alpha$?

- Two person zero sum game.
- Player 1 wins iff $\mathfrak{A} \models \alpha$.

Rules of the evaluation game

Game $G(\alpha, \mathfrak{A})$

- $\alpha \equiv \alpha_1 \vee \alpha_2$: Player 1 chooses a disjunct.
- $\alpha \equiv \alpha_1 \wedge \alpha_2$: Player 2 chooses a conjunct.
- $\alpha \equiv \exists x \alpha_1$: Player 1 chooses an element a from the domain and play continues with $\alpha_1[x := a]$.
- $\alpha \equiv \forall x \alpha_1$: Player 2 chooses an element a from the domain and play continues with $\alpha_1[x := a]$.
- $\alpha \equiv \neg \alpha_1$: Players switch roles.
- α is atomic: Player 1 wins if it is true, Player 2 wins if it is false.

Logical analysis of games

Game operators

- A set of atomic games $(g_1, g_2, \ldots)$.
- Choice - $g_1 \cup g_2$ (disjunction).
- Sequential composition - $g_1; g_2$ $(\exists x \alpha(x))$.
- Dual - g^d (negation).
- Iteration - g^* (fix point operator).

Logical analysis of games

Interpretation

- Final outcomes that players can enforce.
- Set of states S.
- Effectivity relation - $E_g \subseteq S \times 2^S$
  - $(s, X) \in E_g$ iff starting at s, in game g, player 1 can enforce the outcome to be in X.

Game logic [Parikh]

A logic to reason about determined two person zero sum games.

Game logic

Syntax

- $\Phi := p \mid \neg\alpha \mid \alpha_1 \vee \alpha_2 \mid \langle\gamma\rangle\alpha$.
- $\Gamma := g \in \Gamma_0 \mid \gamma_1;\gamma_2 \mid \gamma_1 \cup \gamma_2 \mid \gamma^d \mid \gamma^*$.

Model $M = (S, \{E_g \mid g \in \Gamma_0\}, V)$.

Neighbourhood semantics

- $M, s \models \langle\gamma\rangle\alpha$ iff $\exists(s, X) \in E_\gamma$ such that $X \subseteq \{s' \mid M, s' \models \alpha\}$.
- Player 1 has the ability in game γ to ensure α.

Game logic

- The neighbourhood relation captures what players can enforce in the game.
- Logic reasons about composing neighbourhoods or abilities of players.

Game logic

- The neighbourhood relation captures what players can enforce in the game.
- Logic reasons about composing neighbourhoods or abilities of players.

- Does not specify the mechanism by which the players attain their abilities.
- Abilities of players arise from strategies they have access to.
- Bring in strategy dependence explicitly into the logical reasoning.

Strategic reasoning in compositional games

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System of interacting agents

- Agents interact in various contexts, interaction is not one shot.
- Interactive behaviour modelled as games are compositional.
- Computationally limited agents employing bounded memory strategies.

Strategic reasoning in compositional games

- Classical solutions concepts (Nash equilibrium) does not take into account limitations of players [Halpern,Pass].
  - Players have bounded computational capabilities.
  - Aspects of game like payoffs, actions available need not be common knowledge.
- A player's strategic response can take into account other players' moves in the past.
- In compositional games, history information is available to players.
- Games and choice of players need to be composed together.

- Consider the situation where each atomic game is presented as a normal form game.

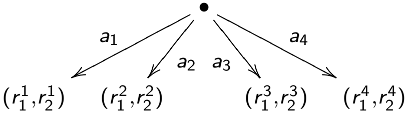
Normal form games

		Player 2	
		x	y
Player 1	b	(r_1^1, r_2^1)	(r_1^2, r_2^2)
	c	(r_1^3, r_2^3)	(r_1^4, r_2^4)

- Rows correspond to strategies of player 1.
- Columns corresponds to strategies of player 2.
- Entries of the matrix - utilities or rewards for players.
- Strategy profile constitutes a play.

Normal form games

	x	y
b	(r_1^1, r_2^1)	(r_1^2, r_2^2)
c	(r_1^3, r_2^3)	(r_1^4, r_2^4)



$$a_1 = (b, x) \quad a_2 = (b, y)$$
$$a_3 = (c, x) \quad a_4 = (c, y)$$

Normal form games

One shot games

- Strategies are presented in an abstract manner.
- Analysis is driven by outcome specification.
  - Rational response, dominant strategy, equilibrium.
- Logical analysis of strategies with respect to outcomes of the game [Bonanno].
  - Capturing utilities in terms of propositions of the logic.

Logical analysis of games

Structured games

- Each state is associated with a single normal form game.
- Alternating time temporal logic [Alur et al].
- Makes assertions on what coalitions of players can enforce.

Logical analysis of games

Strategic reasoning in ATL

- In terms of epistemic conditions of players ([Jamroga, van der Hoek], [van der Hoek, Wooldridge]).
- Logic where (functional) strategies are explicitly part of the formalism ([van der Hoek et al],[Walther et al]).
- Ability to reason about specific actions of players ([Agotnes],[Borgo]).

Logical analysis of games

Compositional games

- Reasoning about abilities of players ([Parikh],[Pauly]).
- Game algebras ([Goranko]).
- Role of strategies is limited to expressing the abilities of players.

The logic

Syntax

- $\Gamma := (g, \eta) \mid \xi_1; \xi_2 \mid \xi_1 \cup \xi_2 \mid \xi^*$

  where η is a set of plays in g.
- $\Phi := p \in P \mid \neg\alpha \mid \alpha_1 \vee \alpha_2 \mid \langle\xi\rangle^{\forall}\alpha$.

Neighbourhood semantics

- $M, u \models \langle\xi\rangle^{\forall}\alpha$ iff $\exists(u, X) \in R_{\xi}$ such that $X \subseteq \{w \mid M, w \models \alpha\}$.

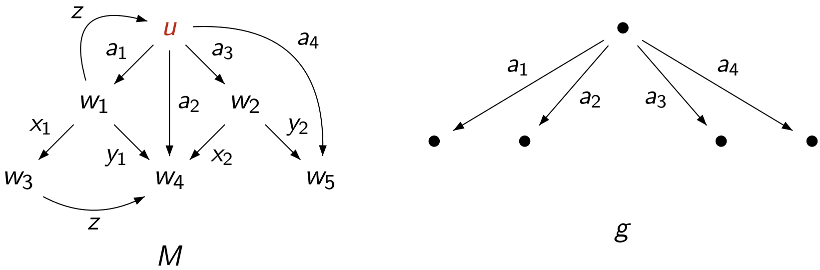
The logic

- How is the neighbourhood relation constructed ?
- In abstract games neighbourhood relation is part of the model.
- When games have structure, neighbourhoods need to be built in a choice constrained manner from the game.
- $R_{(g,\eta)}$ needs to be defined in a way that η restricts g.

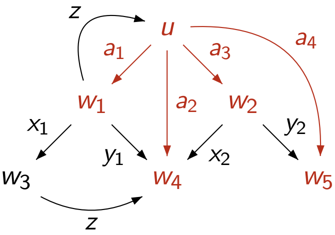
Model

- Model - Kripke structure.
  - A finite set of states W.
  - Labelled edge relation $\longrightarrow \subseteq W \times \Sigma \times W$.
  - Valuation function $V : W \rightarrow 2^P$.
- $(u, X) \in R_{(g,\eta)}$ iff g is enabled at u and X is the set of nodes reachable on plays specified by η.

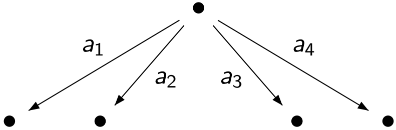
Neighbourhood relation



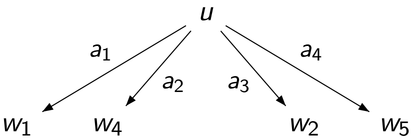
Neighbourhood relation



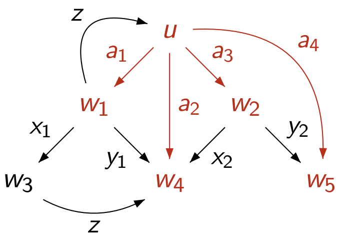
M



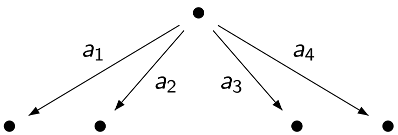
g



Neighbourhood relation



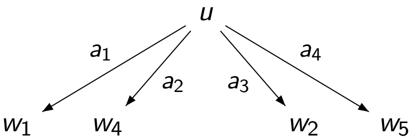
M



g

$\eta = \{a_1, a_2\}$

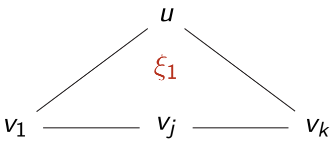
$(u, \{w_1, w_4\}) \in R_{(g,\eta)}$



Semantics of game - strategy pairs

$$\Gamma := (g, \eta) \mid \xi_1 ; \xi_2 \mid \xi_1 \cup \xi_2 \mid \xi^*$$

- $(u, X) \in R_{\xi_1;\xi_2}$



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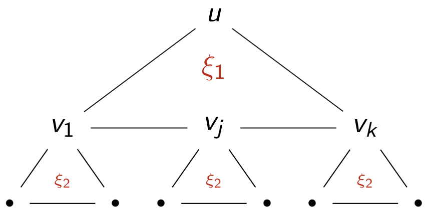

Semantics of game - strategy pairs

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- $(u, X) \in R_{\xi_1;\xi_2}$ iff
  - $\exists Y = \{v_1, \ldots, v_k\}$ such that $(u, Y) \in R_{\xi_1}$.
  - $\forall v_j \in Y$, $\exists X_j \in X$ such that $(v_j, X_j) \in R_{\xi_2}$.
  - $X = \bigcup_{j=1,\ldots,k} X_j$.

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  - $X = \bigcup_{j=1,\ldots,k} X_j$.
- $R_{\xi_1 \cup \xi_2} = R_{\xi_1} \cup R_{\xi_2}$.
- $R_{\xi^*} = \bigcup_{n \geq 0} (R_\xi)^n$.

Examples

- $\langle (g, \eta_2); (g', \eta_1) \rangle^{\forall} \alpha$
  - η_2 - strategy of player 2.
  - η_1 - strategy of player 1.

  Assuming in game g, player 2 employs strategy η_2, player 1 can follow η_1 in game g' to ensure α.

- Not equivalent to: Player 1 can ensure α in the composed game $g = g; g'$.

Examples

- $\langle((g_1, \eta_2); (g_2, \eta_3))^*; p?; (g, \eta_1)\rangle^{\forall}\alpha$.
  - η_2 and η_3 - strategies of player 2.
  - η_1 strategy of player 1.

If player 2 can ensure p by iterating the structure g_1 followed by g_2 employing strategies η_2 and η_3 then player 1 plays according to η_1 is game g.

Truth checking

Problem: Given a model M, a state u and a formula α, check if $M, u \models \alpha$.

- For each composite game - play pair ξ, construct R_ξ.
- Enumerate subformulas of α in the increasing order of complexity and label the states.
- Use two labelling functions.

Labelling 1 (l_1):

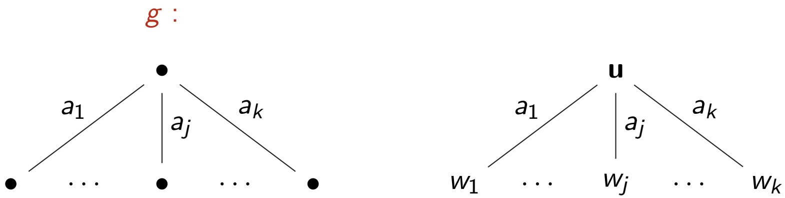
Labels each state with subformulas of α.

Labelling 2 (l_2):

Labels each pair (u, ξ) with a set of subsets of W.

Truth checking

- Question: For a state u and a pair $\xi = (g, \eta)$ when is $X \in l_2(u, \xi)$?



$$\eta \subseteq \{a_1, \ldots, a_k\}$$

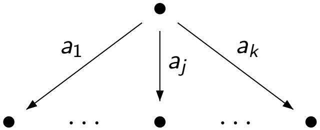
- $X \in l_2(u, (g, \eta))$ where $X = \{w \mid u \xrightarrow{a} w, a \in \eta\}$.

Axiom system

- $\langle g, \eta \rangle^{\forall} \alpha \equiv$?

  Game g is enabled and α holds at all states reachable on plays specified by η.

Axiom system



- g is enabled: $g^{\checkmark} \equiv \bigwedge_{j=1,\ldots,k} \langle a_j \rangle \mathit{True}$.

Axiom:

- $\langle g, \eta \rangle^{\forall} \alpha \equiv g^{\checkmark} \wedge (\bigwedge_{a \in \eta} [a]\alpha)$.

Axiom system

Axioms:

- All the substitutional instances of tautologies of PC.
- $\langle a \rangle(\alpha_1 \vee \alpha_2) \equiv \langle a \rangle \alpha_1 \vee \langle a \rangle \alpha_2$.
- Dynamic logic axioms:
  - $\langle \xi_1 \cup \xi_2 \rangle^{\forall} \alpha \equiv \langle \xi_1 \rangle^{\forall} \alpha \vee \langle \xi_2 \rangle^{\forall} \alpha$.
  - $\langle \xi_1 ; \xi_2 \rangle^{\forall} \alpha \equiv \langle \xi_1 \rangle^{\forall} \langle \xi_2 \rangle^{\forall} \alpha$.
  - $\langle \xi^* \rangle^{\forall} \alpha \equiv \alpha \vee \langle \xi \rangle^{\forall} \langle \xi^* \rangle^{\forall} \alpha$.

Inference rules

$$(MP)\ \frac{\alpha, \quad \alpha \supset \beta}{\beta} \qquad (NG)\ \frac{\alpha}{[a]\alpha}$$

$$(IND)\ \frac{\langle \xi \rangle^{\forall} \alpha \supset \alpha}{\langle \xi^* \rangle^{\forall} \alpha \supset \alpha}$$

Conclusion

Summary

- Proposed a dynamic logic to reason about compositional games and strategies.
- Strategic response of players in the compositional framework.

Remarks

- Enrich the framework to deal with expectation of players.
- How does the notion of composition of games relate to that of sub-games.
- Verifying properties of multi-agent systems.