Monk algebras, completions, and atom structures

Robin Hirsch and Ian Hodkinson

Aim

To show that relation algebras can shed light on duality-related matters.

To nail more nails in the coffin of the hope that **RRA** is docile.

But also perhaps, to show there is value in proving more ‘negative’ results about **RRA**.

Outline

1. We define some relation algebras (like ‘Monk algebras’), based on graphs.
2. We use them to show that **RRA** is not closed under completions (which we will define).
3. If time, we use them to study the atom structures of atomic relation algebras.

0. Relation algebra atom structures, complex algebras (revision)

Abstractly, a *relation algebra atom structure* is a structure $\mathcal{S} = (S, Id, \breve{}, C)$ satisfying certain first-order conditions (the 'correspondents' of Tarski's RA axioms).

If $\mathcal{A}$ is an atomic relation algebra then $\mathrm{At}\,\mathcal{A}$ is such a structure.

Conversely, given such an $\mathcal{S}$, we can form its *complex algebra:*

$$\mathcal{S}^+ = (\wp(S), \cup, ^-, \emptyset, S, 1', \breve{}, ;),$$

where

- $1' = \{x \in \mathcal{S} : \mathcal{S} \models Id(x)\}$
- $\breve{a} = \{\breve{x} : x \in a\}$ (for $a \subseteq S$)
- $a\,;b = \{z \in S : \exists x \in a\ \exists y \in b(\mathcal{S} \models C(x, y, z))\}$ (for $a, b \subseteq S$).

$\mathcal{S}^+$ is an atomic relation algebra, and $\mathrm{At}\,\mathcal{S}^+ \cong \mathcal{S}$.
If $\mathcal{A}$ is an atomic relation algebra, $\mathcal{A} \hookrightarrow (\mathrm{At}\,\mathcal{A})^+$.

1. Graphs and relation algebras

Here, graphs are undirected: $G = (V, E)$ where $E \subseteq V \times V$ is symmetric.
They can have loops ($(x, x) \in E$), but they won't in this talk.

A subset $X \subseteq V$ is *independent* if $E \cap (X \times X) = \emptyset$.

For $k < \omega$, a *k-colouring* of G is a partition of V into $\leq k$ independent sets.

The *chromatic number $\chi(G)$* of G is the least $k < \omega$ such that G has a k-colouring, and ∞ if there is no such k.

A *cycle of length k* in a graph G is a sequence $v_1, \ldots, v_k$ of distinct nodes of G, such that $(v_1, v_2), \ldots, (v_{k-1}, v_k), (v_k, v_1)$ are edges of G.

Fact 1 *$\chi(G) \leq 2$ iff G has no cycles of odd length.*

From now on, we write $x \in G$, $X \subseteq G$, instead of $x \in V$, $X \subseteq V$, etc.

Relation algebras from graphs

Fix a graph G. We define a relation algebra $\mathcal{A}(G)$.
Its set of atoms is $Q = \{1', \mathsf{r}_x, \mathsf{b}_x, \mathsf{g}_x : x \in G\}$ — red, blue and green copies of nodes of G. $\mathcal{A}(G)$ is the full complex algebra over Q:

$$\mathcal{A}(G) = (\wp(Q), \cup, -, \emptyset, Q, \{1'\}, \breve{}, ;).$$

We define $\breve{S} = S$, for all $S \subseteq Q$.
We define composition by listing the *inconsistent triples of atoms* — those (a, b, c) such that $(a ; b) \cdot c = 0$. They are:

- $(1', a, b)$, $(a, 1', b)$, and $(a, b, 1')$, whenever $a \neq b$,
- $(\mathsf{r}_x, \mathsf{r}_y, \mathsf{r}_z)$, $(\mathsf{b}_x, \mathsf{b}_y, \mathsf{b}_z)$, and $(\mathsf{g}_x, \mathsf{g}_y, \mathsf{g}_z)$, whenever $\{x, y, z\}$ *is independent in* G.

Fact: $\mathcal{A}(G)$ is a simple relation algebra (a kind of 'Monk algebra').

Independent elements of $\mathcal{A}(G)$

For $X \subseteq G$, define $\mathsf{R}_X = \{\mathsf{r}_x : x \in X\} \in \mathcal{A}(G)$.
Define $\mathsf{B}_X, \mathsf{G}_X$ similarly.

An element $a \in \mathcal{A}(G)$ is said to be *independent* if $a = \mathsf{C}_X$ for some $\mathsf{C} \in \{\mathsf{R}, \mathsf{B}, \mathsf{G}\}$ and some independent $X \subseteq G$.

Lemma 2 *Let $a \in \mathcal{A}(G)$.*

1. *If a is independent, then $(a\,;a) \cdot a = 0$.*

2. *If $a \leq -1'$ is not independent, then $a\,;a = 1$.*

Chromatic number and representability

Let $\mathcal{B} \subseteq \mathcal{A}(G)$ be a subalgebra. $\mathcal{B}$ is said to be *balanced* if

- $\mathsf{R}_G, \mathsf{B}_G, \mathsf{G}_G \in \mathcal{B}$,
- for all $X \subseteq G$, $\mathsf{R}_X \in \mathcal{B} \iff \mathsf{B}_X \in \mathcal{B} \iff \mathsf{G}_X \in \mathcal{B}$.

The *chromatic number* $\chi(\mathcal{B})$ of a balanced $\mathcal{B} \subseteq \mathcal{A}(G)$ is the least $k < \omega$ such that R_G is the sum of k independent elements of $\mathcal{B}$, and ∞ if there is no such k.

Remark: $\chi(G) = \chi(\mathcal{A}(G)) \leq \chi(\mathcal{B})$.

Theorem 3 *For infinite balanced $\mathcal{B} \subseteq \mathcal{A}(G)$, we have*

$$\mathcal{B} \in \textbf{RRA} \iff \chi(\mathcal{B}) = \infty.$$

Proof: $\mathcal{B}$ is representable $\Rightarrow \chi(\mathcal{B}) = \infty$

$\mathcal{B}$ is simple. Suppose that $h : \mathcal{B} \to \mathfrak{Re}(U)$ is a representation. Because $\mathcal{B}$ is infinite, so is U. Pick distinct $x_0, x_1, \ldots \in U$.

Assume for contradiction that $\chi(\mathcal{B}) < \infty$. So '$-1$' $= \sum_{i \leq n} b_i$, for some independent elements $b_1, \ldots, b_n \in \mathcal{B}$.

h respects '-1'. So for all $i < j$, we have $(x_i, x_j) \in h(\sum_{i \leq n} b_i)$. h respects $+$. So there is $b^{ij} \in \{b_1, \ldots, b_n\}$ with $(x_i, x_j) \in h(b^{ij})$.

By Ramsey's theorem, we can assume b^{ij} is constant — say, $b^{ij} = b_7$ for all $i < j$.

Triangle consistency for (x_0, x_1, x_2) gives $(b_{01} \,;\, b_{12}) \cdot b_{02} \neq 0$. That is,

$$(b_7 \,;\, b_7) \cdot b_7 \neq 0.$$

But b_7 is independent, so by lemma 2, $(b_7 \,;\, b_7) \cdot b_7 = 0$. Contradiction!

Proof: $\chi(\mathcal{B}) = \infty \quad \Rightarrow \quad \mathcal{B}$ is representable

Assume $\chi(\mathcal{B}) = \infty$.

Say $b \in \mathcal{B}$ is R-*big* if $\mathsf{R}_G - b$ is the sum of finitely many independent elements of $\mathcal{B}$.

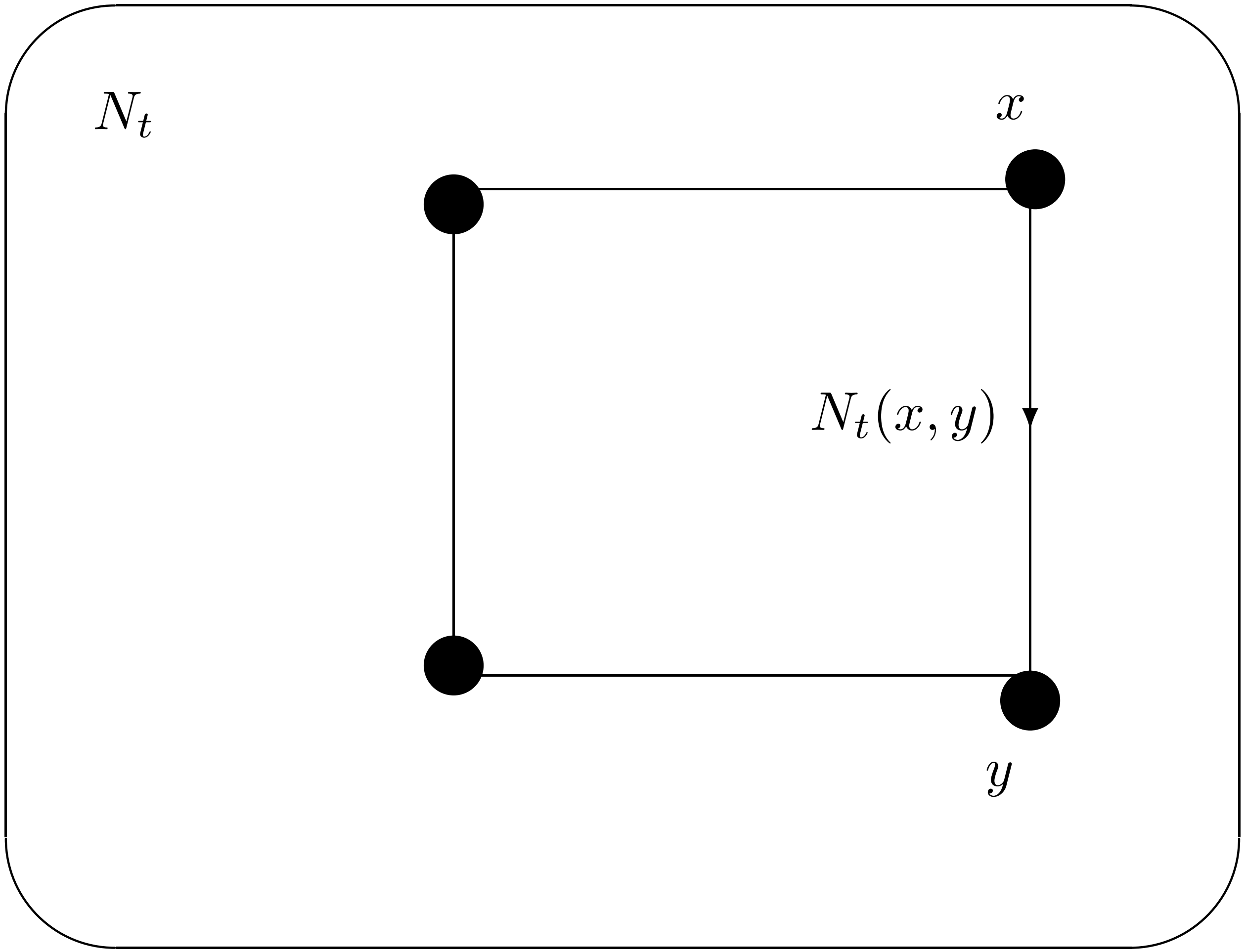
The set of R-big elements of $\mathcal{B}$ has the finite intersection property. So (using some form of AC) it extends to an ultrafilter R_μ of $\mathcal{B}$.

$\mathsf{R}_G \in \mathsf{R}_\mu$. If $a \in \mathcal{B}$ is independent, $-a$ is R-big, so $-a \in \mathsf{R}_\mu$, so $a \notin \mathsf{R}_\mu$. So no element of R_μ is independent. Hence, by lemma 2,

$$r \,;\, r' = 1 \;\text{ for any } r, r' \in \mathsf{R}_\mu.$$

Choose ultrafilters $\mathsf{B}_\mu, \mathsf{G}_\mu$ similarly.

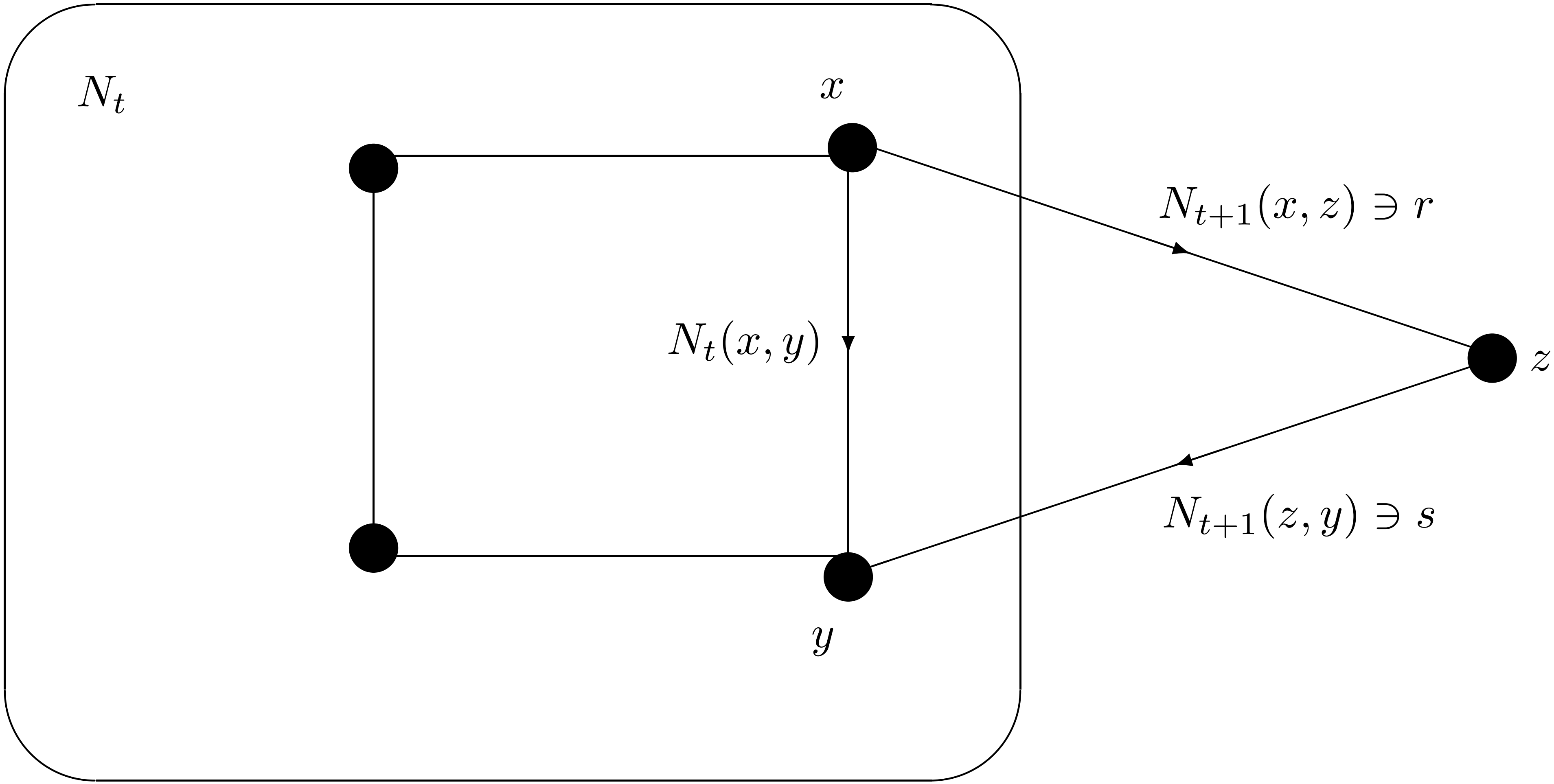
We now show that $\exists$ can use these ultrafilters to win the game $G^u_\omega(\mathcal{B})$ of length ω played on ultrafilter networks over $\mathcal{B}$. So $\mathcal{B} \in$ **RRA**.



$\forall$ picks $x, y \in N_t$ and $r, s \in \mathcal{B}$ with $r\,;s \in N_t(x, y)$.

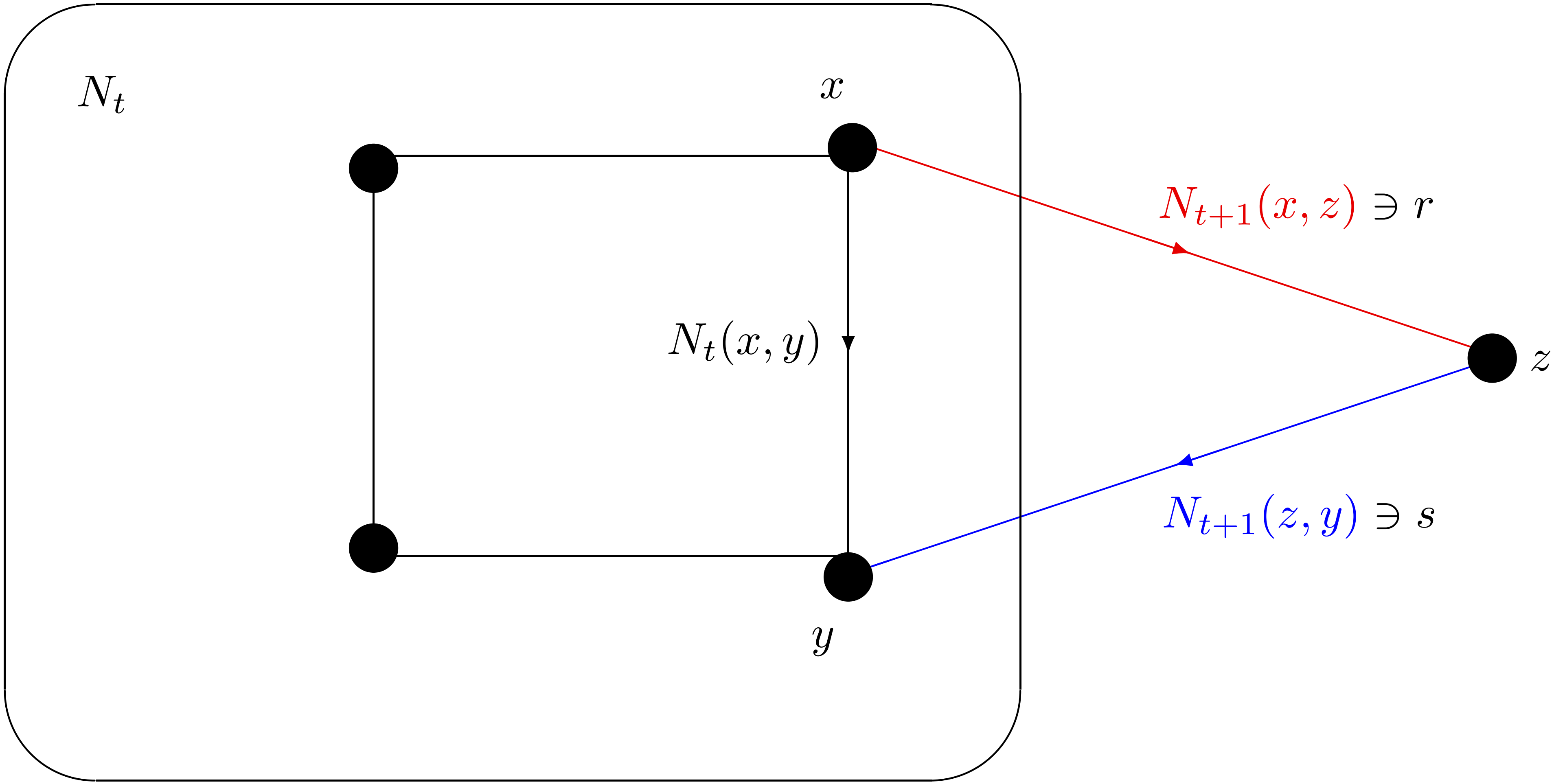
We assume no existing witness in N_t.

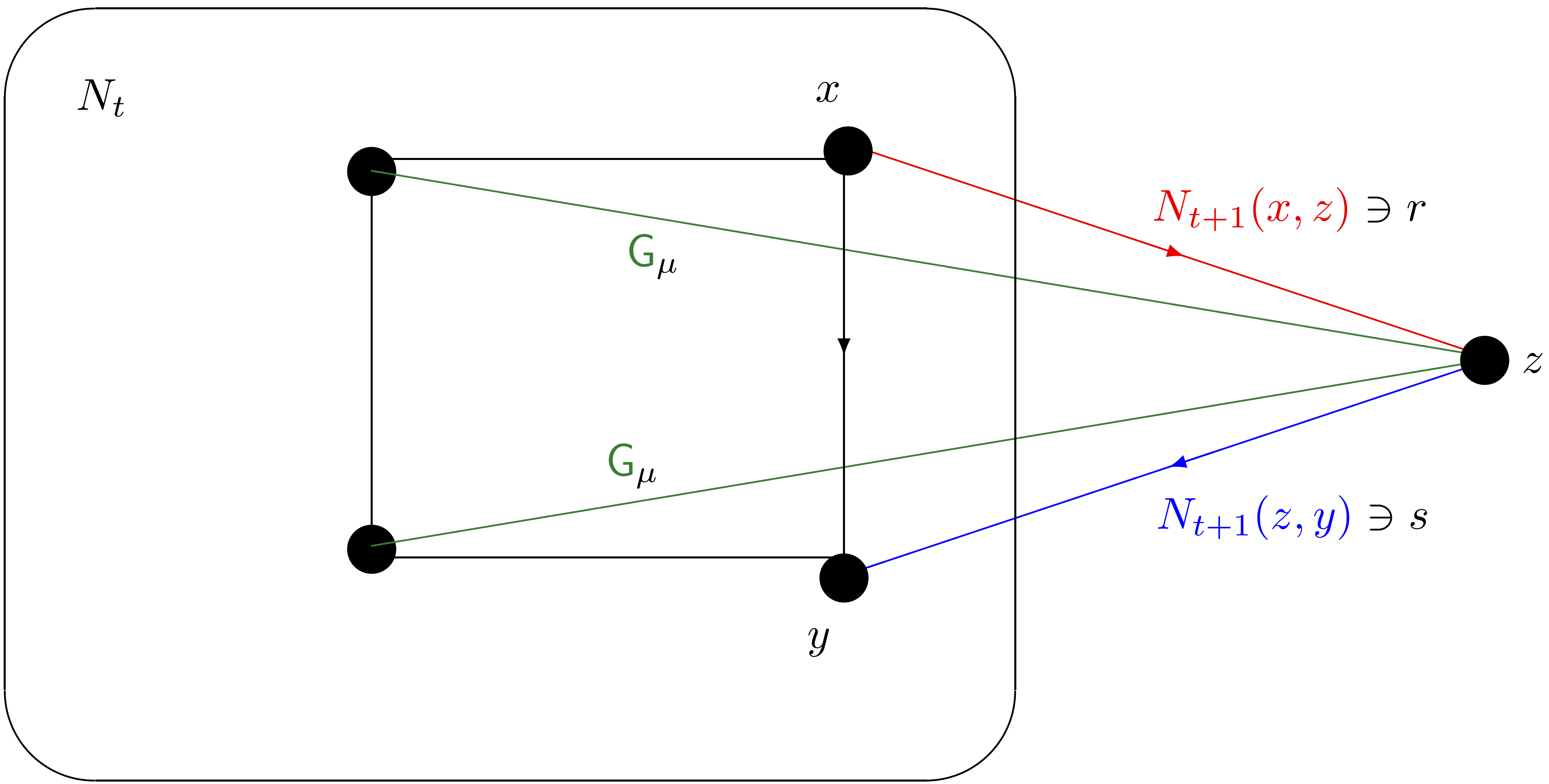
$\exists$ chooses ultrafilters $N_{t+1}(x,z) \ni r$ and $N_{t+1}(z,y) \ni s$
with $N_{t+1}(x,z)\,;N_{t+1}(z,y) \subseteq N_t(x,y)$.



$R_G + B_G + G_G \in N_{t+1}(x, z)$. Say, $R_G \in N_{t+1}(x, z)$.

Similarly, say $B_G \in N_{t+1}(z, y)$.




N_t
x
$N_{t+1}(x, z) \ni r$
G_μ
z
G_μ
$N_{t+1}(z, y) \ni s$
y

Summing up

We have proved that for infinite balanced $\mathcal{B} \subseteq \mathcal{A}(G)$,

$$\mathcal{B} \in \textbf{RRA} \iff \chi(\mathcal{B}) = \infty.$$

In the rest of the talk, we will see some applications of this.

2. Completions

A *completion* of a relation algebra $\mathcal{A}$ is a relation algebra $\overline{\mathcal{A}}$ such that

1. $\mathcal{A} \subseteq \overline{\mathcal{A}}$,
2. $\overline{\mathcal{A}}$ is complete as a boolean algebra: $\sum S$ exists for all $S \subseteq \overline{\mathcal{A}}$,
3. $\mathcal{A}$ is dense in $\overline{\mathcal{A}}$: $\forall c \in \overline{\mathcal{A}} \setminus \{0\}\ \exists a \in \mathcal{A} \setminus \{0\}\ (a \leq c)$.

Monk (1970): every relation algebra has a completion, which is unique up to isomorphism.

Easy fact: If $\mathcal{B}$ is an atomic relation algebra, then its completion is $(\mathrm{At}\,\mathcal{B})^+$ — the complex algebra over its atom structure.

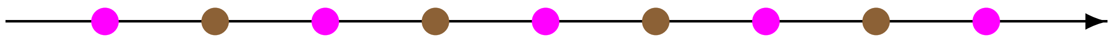
Theorem (IH, 1997; this proof by R Hirsch–IH 2002)

Theorem 4 **RRA** *is not closed under completions.*

This answers an implicit question of Monk (1970).

Proof. Let $G = (\mathbb{Z}, E)$ where $(x, y) \in E$ iff $|x - y| = 1$.

Then $\chi(\mathcal{A}(G)) = \chi(G) = 2$.



Let $\mathcal{B}$ be the subalgebra of $\mathcal{A}(G)$ whose elements are finite sums (unions) of $\{1'\}$ and $\mathsf{R}_X, \mathsf{B}_X, \mathsf{G}_X$ for all finite or cofinite $X \subseteq \mathbb{Z}$.

Can check this *is* a subalgebra. It is infinite and balanced.

Then $\chi(\mathcal{B}) = \infty$. So $\mathcal{B} \in$ **RRA**.

$\mathcal{A}(G)$ is a completion of $\mathcal{B}$.

But $\chi(\mathcal{A}(G)) = 2$. So $\mathcal{A}(G) \notin$ **RRA**. ■

RRA is not Sahlqvist-axiomatisable

Sahlqvist equations are defined syntactically. E,g., all positive equations are Sahlqvist.

Corollary 5 (Venema, 1997) **RRA** *is not axiomatisable by Sahlqvist equations.*

Proof. By [Givant–Venema 1999], Sahlqvist equations are preserved under completions of 'conjugated BAOs' (e.g., relation algebras).

By theorem 4, **RRA** is not closed under completions. ■

3. Weakly and strongly representable atom structures

Representability of an atomic relation algebra is not determined by its atom structure. There are two atomic relation algebras ($\mathcal{B}$ and $\mathcal{A}(G)$) with the *same* atom structure, one representable, the other not.

So let's distinguish two kinds of relation algebra atom structure $\mathcal{S}$:

- *weakly representable:* some atomic relation algebra with atom structure $\mathcal{S}$ is representable,
- *strongly representable:* every atomic relation algebra with atom structure $\mathcal{S}$ is representable.

Theorem 6 (Venema, 1998) *The class of* weakly representable *relation algebra atom structures is elementary.*

(He proved a general result for varieties of completely additive BAOs.)

What about the strongly representable atom structures?

Write **SRAS** for the class of strongly representable atom structures.

Theorem 7 (Hirsch–IH, 2002) SRAS *is not elementary.*

The proof uses

- a simple lemma on strongly representable atom structures
- the 'Monk algebras' $\mathcal{A}(G)$
- Erdős graphs
- ultraproducts

Simple lemma

Lemma 8 *A relation algebra atom structure $\mathcal{S}$ is strongly representable iff $\mathcal{S}^+ \in$* **RRA***.*

Proof. TFAE:

1. $\mathcal{S}$ is strongly representable
2. every atomic relation algebra $\mathcal{A}$ with atom structure $\mathcal{S}$ is representable
3. $\mathcal{S}^+$ is representable.

   We said yesterday that for any atomic relation algebra $\mathcal{A}$,

$$\begin{array}{rcl} \mathcal{A} & \hookrightarrow & (\mathrm{At}\,\mathcal{A})^+ \\ a & \mapsto & \{x \in \mathrm{At}\,\mathcal{A} : x \leq a\}. \end{array}$$

■

Erdős graphs

Theorem 9 (Erdős, 1959) *For any finite k, there is a finite graph E_k of chromatic number $> k$ and with no cycles of length $< k$.*

For each $n < \omega$, let $G_n = \dot{\bigcup}_{k \geq n} E_k$ (an infinite graph).

Then

- $\chi(G_n) = \infty$.
  So by theorem 3, $\mathcal{A}(G_n) \in$ **RRA**.
- Notation: for a graph G, write $\alpha(G)$ for $\mathrm{At}\,\mathcal{A}(G)$. So $\mathcal{A}(G) = \alpha(G)^+$.

  By our simple lemma 8, $\alpha(G_n) \in$ **SRAS**.
- Also, G_n has no cycles of length $< n$.

SRAS not closed under ultraproducts

Now take a non-principal ultrafilter D over ω.

1. $\prod_D G_n$ has no cycles (Łoś's theorem).
   So $\chi(\prod_D G_n) \leq 2$.
   By theorem 3, $\mathcal{A}(\prod_D G_n) \notin$ **RRA**.

2. Let $\mathcal{S} = \prod_D \alpha(G_n)$.
   $\mathcal{S} \cong \alpha(\prod_D G_n)$ (exercise)
   By (1) and simple lemma 8, $\mathcal{S} \notin$ **SRAS**.

Conclude: $\alpha(G_n) \in$ **SRAS** for all n, but an ultraproduct ($\mathcal{S}$) of the $\alpha(G_n)$ is not in **SRAS**.

Therefore, **SRAS** is not closed under ultraproducts, so (by Łoś's theorem) is not elementary.

Summary

Venema: the class of weakly representable relation algebra atom structures is elementary.

We proved that the class of strongly representable relation algebra atom structures is not elementary.

This answers a question of Maddux (1982).

Conclusion

The $\mathcal{A}(G)$ are variants of *Monk algebras.* Simple and useful:

- **RRA** not closed under Monk completions, not Sahlqvist-ax'ble
- **SRAS** non-elementary (& similarly for strongly representable atom structures of cylindric algebras)
- IH–Venema, 2002: Any first-order (e.g., equational) axiomatisation of **RRA** has infinitely many 'non-canonical' sentences (also true for McKinsey–Lemmon modal logic).
- Goldblatt–IH–Venema, 2003: There are canonical modal logics that are not the logic of any elementary class of frames.
  This answers a question of Fine (1975).

Binary relations, and representations of RAs, are fundamental. The list suggests there is value in proving 'negative' results about them.

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