Substructural Logics

Algebraic view of logical properties

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We talk about:

- Algebraic approach to cut elimination
- Interpolation properties and Amalgamation properties

Cut elimination

- Gentzen's original proof gives a procedure how to eliminate each application of cut rule in a given proof. It needs double induction.
- Each application of cut rule can be eliminated, depending on where and how it appears.
- It should be noted that cut rule cannot be replaced by a (uniform) combination of applications of other rules.

Syntactic proofs of cut elimination are quite informative, as they analyze structures of proofs directly.

Semantical cut elimination

Semantical proofs of cut elimination theorem

- K. Schütte, 1960 — semi valuation,
- J.-Y. Girard — three-valued semantics,
- M. Fitting, 1973 — consistency property for constructing Kripke models.

Algebraic cut elimination

- Several attempts have been made to show cut elimination in an algebraic way, e.g. Maehara (1991), Okada (1996) and Jipsen-Tsinakis (2002).
- Our proof in (BJO) is purely algebraic, inspired by these works.
- Sometimes algebraic proofs will be more flexible and can provide a wider view.

(BJO) *F. Belardinelli, P. Jipsen and H. Ono, Algebraic aspects of cut elimination, Studia Logica 77 (2004), 209-240.*

Why algebraic proofs?

- to clarify meaning of cut elimination from algebraic point of view,
- to give a proof of cut elimination comprehensible for algebraists, avoiding heavy syntactic arguments.

Algebraic cut elimination

Algebraic proof of cut elimination by BJO.

- For each sequent system $\mathcal{S}_L$ of a logic $\mathbf{L}$, partial structures, called Gentzen matrices for $\mathcal{S}_L$ *without cut*, are introduced.
- It is shown that each Gentzen matrix $\mathbf{Q}$ for $\mathcal{S}_L$ is quasi-embeddable into a complete algebra $\mathbf{B}$ for $\mathbf{L}$, called a quasi completion of $\mathbf{Q}$.

Cut elimination for $\mathbf{FL_{ew}}$

We will give an algebraic proof of cut elimination for $\mathbf{FL_{ew}}$, i.e., $\mathbf{FL}$ with both exchange and weakening rules.

Cut elimination for $\mathbf{FL_{ew}}$ says:

- If a sequent $\Gamma \Rightarrow \delta$ is provable in $\mathbf{FL_{ew}}$ then it is provable in $\mathbf{FL_{ew}}$ without using cut rule.

Recall that cut rule is:

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Sigma \Rightarrow \delta}{\Gamma, \Sigma \Rightarrow \delta}$$

$\mathbf{FL_{ew}}$-algebras

An algebra $\mathbf{A} = \langle A; \wedge, \vee, \cdot, \rightarrow, 1, 0\rangle$ is an $\mathbf{FL_{ew}}$-algebras iff

- $\langle A; \wedge, \vee, 1, 0\rangle$ is a bounded lattice,
- $\langle A; \cdot, 1\rangle$ is a commutative monoid with the unit 1,
- $x \cdot y \leq z$ iff $x \leq (y \rightarrow z)$ (the law of residuation)

Hence, any $\mathbf{FL_{ew}}$-algebra is a commutative $\mathbf{FL}$-algebra such that the unit element 1 is the greatest element and 0 is the least, respectively.

$\mathbf{FL_{ew}}$-algebras: completeness

A sequent $\alpha, \beta, \ldots, \gamma \Rightarrow \delta$ is valid in an $\mathbf{FL_{ew}}$-algebra $\mathbf{A}$ iff

$$\mathbf{A} \models \alpha \cdot \beta \cdots \gamma \leq \delta,$$

i.e.

$h(\alpha) \cdot h(\beta) \cdots h(\gamma) \leq h(\delta)$ holds in $\mathbf{A}$ for any assignment h.

Completeness of $\mathbf{FL_{ew}}$ with respect to $\mathbf{FL_{ew}}$-algebras.

A sequent $\alpha, \beta, \ldots, \gamma \Rightarrow \delta$ *is provable in* $\mathbf{FL_{ew}}$ *iff it is valid in every* $\mathbf{FL_{ew}}$*-algebra.*

Some observations

- Without cut rule, "commas" in sequents cannot be replaced by fusions $\cdot$.
- Cut rule preserves the validity in every $\mathbf{FL_{ew}}$-algebra, because the following holds in it.

  $x \leq a$ and $a \cdot y \leq z$ imply $x \cdot y \leq z$,

  which is equivalent to the following two:
  - $\leq$ is transitive,
  - $a \leq b$ implies $a \cdot y \leq b \cdot y$.

A Gentzen matrix for $\mathbf{FL_{ew}}$ is a relational structure for $\mathbf{FL_{ew}}$ without cut.

Gentzen matrices

For a nonempty set B,

B^* is the set of all multisets of members of B,

which forms a commutative monoid with respect to the multiset union, with the unit ϵ, i.e. the empty multiset.

Letters x, y, z, u, v (a, b, c, d) are used for elements in B^* (in $B \cup \{\epsilon\}$,resp.). xy denotes the multiset union of x and y.

A Gentzen matrix for $\mathbf{FL_{ew}}$ is a relational structure $\mathbf{B} = \langle B; \wedge, \vee, \cdot, \rightarrow; \preceq \rangle$, where $\wedge, \vee, \cdot, \rightarrow$ are binary operations on B, and $\preceq$ is a subset of $B^* \times (B \cup \{\epsilon\})$ such that:

Gentzen matrices

- $a \preceq a$,
- $x \preceq c$ implies $dx \preceq c$,
- $x \preceq a$ and $by \preceq c$ imply $(a \to b)xy \preceq c$,
- $ax \preceq b$ implies $x \preceq a \to b$,
- $ax \preceq c$ and $bx \preceq c$ imply $(a \vee b)x \preceq c$,
- $x \preceq a$ implies $x \preceq a \vee b$,
- $x \preceq b$ implies $x \preceq a \vee b$,
- $ax \preceq c$ implies $(a \wedge b)x \preceq c$,
- $bx \preceq c$ implies $(a \wedge b)x \preceq c$
- $x \preceq a$ and $x \preceq b$ imply $x \preceq a \wedge b$,
- $abx \preceq c$ implies $(a \cdot b)x \preceq c$,
- $x \preceq a$ and $y \preceq b$ imply $xy \preceq a \cdot b$.

Gentzen matrices

and also:

- $xy \preceq c$ implies $x1y \preceq c$,
- $\epsilon \preceq 1$,
- $0 \preceq \epsilon$,
- $x \preceq \epsilon$ implies $x \preceq 0$.

Each of these conditions comes either from an intial sequent, or from a rule of inference (except cut) of $\mathbf{FL_{ew}}$, by replacing $\Rightarrow$ by $\preceq$.

Gentzen matrices: completeness

A sequent $\alpha, \beta, \ldots, \gamma \Rightarrow \delta$ is valid in a Gentzen matrix $\mathbf{B}$ for $\mathbf{FL_{ew}}$ iff

$$\langle g(\alpha), g(\beta), \cdots, g(\gamma) \rangle \preceq g(\delta)$$

holds in $\mathbf{B}$ for any assignment g.

Completeness of $\mathbf{FL_{ew}}$ without cut with respect to Gentzen matrices:

> *A sequent $\alpha, \beta, \ldots, \gamma \Rightarrow \delta$ is provable in $\mathbf{FL_{ew}}$ without cut iff it is valid in every Gentzen matrix for $\mathbf{FL_{ew}}$.*

Towards cut elimination

To show cut elimination for $\mathbf{FL_{ew}}$, it enough to show that

- *if* $\langle g(\alpha), g(\beta), \cdots, g(\gamma)\rangle \preceq g(\delta)$ *fails for some* g *in a Gentzen matrix* $\mathbf{B}$*, then* $h(\alpha)\cdot h(\beta)\cdots h(\gamma) \leq h(\delta)$ *fails for some* h *in an* $\mathbf{FL_{ew}}$*-algebra* $\mathbf{A}$*.*

How to get such an $\mathbf{FL_{ew}}$-algebra $\mathbf{A}$ from a given Gentzen matrix $\mathbf{B}$?

Quasi-completions

In order to preserve the non-validity of a given seqiuent, $\mathbf{B}$ must be embedded somehow into the algebra $\mathbf{A}$,

1. There is a uniform way of constructing such an $\mathbf{A}$, which is called the quasi-completion of $\mathbf{B}$.
2. In fact, $\mathbf{A}$ is a complete $\mathbf{FL_{ew}}$-algebra of the form $\mathbf{C_{B^*}}$. Here, C is a nucleus on $\wp(B^*)$ and $\mathbf{C_{B^*}}$ is the algebra consisting of all C-closed elements.
3. Furthermore, it is shown that $\mathbf{B}$ is quasi-embedded into $\mathbf{A}$.

Quasi-completions

Each $\mathbf{FL_{ew}}$-algebra can be regarded as a Gentzen matrix satisfying the transitivity if we define $\preceq$ as follows:

$$\langle a_1, \ldots, a_m \rangle \preceq b \ \text{ iff } \ (a_1 \cdot \ldots \cdot a_m) \leq b.$$

When $\mathbf{B}$ is an $\mathbf{FL_{ew}}$-algebra, the quasi-completion of $\mathbf{B}$ (as a Gentzen matrix) is isomorphic to the MacNeille completion of $\mathbf{B}$, and the quasi-embedding becomes complete embedding.

Cut elimination and completion

Our algebraic proof works well not only for many of standard sequent systems of substructural logics and modal logics. The idea can be applied also to completeness proofs of tableaux systems.

From the last observation, it follows that when our approach works for a sequent system for a logic $\mathbf{L}$, the corresponding variety must be closed under the MacNeille completion.

- Note that only three varieties of Heyting algebras are closed under the MacNeille completion.

Finite model property

Different from modal logics, it is not easy to show the FMP of substructural logics. So, the FMP is not a useful property in proving the decidability (cf. Harrop).

In fact, the situation is a bit twisted. Combining quasi-completions with the ideas by Lafont, Okada etc. we can show that:

- *if every proof search ends in finitely many steps in a cut-free system for a logic* $\mathbf{L}$ *(and hence* $\mathbf{L}$ *is in fact decidable), it has the FMP.*

Interpolation properties

This is a brief survey of my joint work (KO) with H. Kihara. Our results can be easily extended to a much wider class of logics.

(KO) *H. Kihara and H. Ono, Interpolation properties, Beth definability properties and amalgamation properties for substructural logics, 53 pages, to appear in Journal of Logic and Computation*.

Some of related literatures are:

- J. Czelakowski and D. Pigozzi, *Amalgamation and interpolation in abstract algebraic logic*, in Lecture Notes in Pure and Applied Math. 203, 1999.
- D. Gabbay and L.L. Maksimova, *Interpolation and Definability, Modal and Intuitionistic Logics*, 2005.

Craig interpolation property

A substructural logic $\mathbf{L}$ has the Craig interpolation property (CIP) if for all formulas ϕ and ψ,

if $\vdash_{\mathbf{L}} \phi\backslash\psi$, then there exists a formula δ such that

- $\vdash_{\mathbf{L}} \phi\backslash\delta$ and $\vdash_{\mathbf{L}} \delta\backslash\psi$,
- $Var(\delta) \subseteq Var(\phi) \cap Var(\psi)$,

where $Var(\gamma)$ denotes the set of all variables in a formula γ. The formula δ is called an interpolant of $\phi\backslash\psi$.

Maehara's method

S. Maehara noticed that the CIP of $\mathbf{L}$ follows from cut elimination of a sequent system for $\mathbf{L}$.

In fact, by using induction of the length of a cut-free proof of a sequent $\Pi \Rightarrow \psi$, the existence of an interpolant δ is shown for any *partition* $\langle \Gamma, \Sigma \rangle$ of Π satisfying that:

- both $\Gamma \Rightarrow \delta$ and $\delta, \Sigma \Rightarrow \psi$ are provable,
- $Var(\delta) \subseteq Var(\Gamma) \cap Var(\Sigma, \psi)$.

Deductive interpolation property

Another interpolation property in the deductive form:

A substructural logic $\mathbf{L}$ has the deductive interpolation property (DIP), if for any set Γ of formulas and for any formula ψ,

if $\Gamma \vdash_{\mathbf{L}} \psi$, then there exists a formula δ such that

- $\Gamma \vdash_{\mathbf{L}} \delta$ and $\delta \vdash_{\mathbf{L}} \psi$,
- $Var(\delta) \subseteq Var(\Gamma) \cap Var(\psi)$,

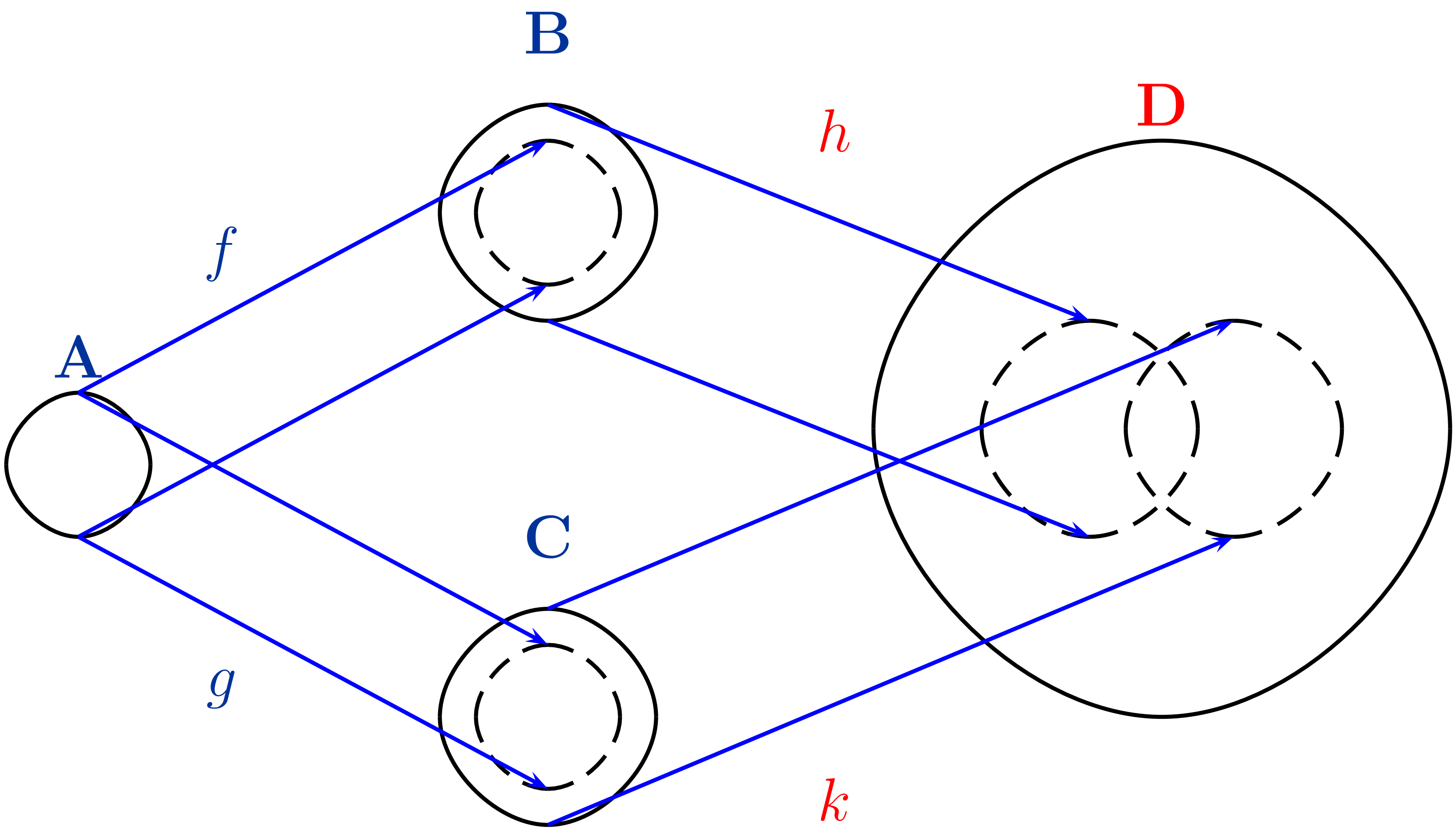
Amalgamation property

A variety $\mathbf{V}$ of FL-algebras has the amalgamation property (AP) , if for all $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathbf{V}$ and for all embeddings $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$

- there exist an algebra $\mathbf{D}$ in $\mathbf{V}$ and embeddings $h : \mathbf{B} \to \mathbf{D}$ and $k : \mathbf{C} \to \mathbf{D}$ such that

$$h \circ f = k \circ g.$$

Amalgamation property



$$h \circ f = k \circ g$$

CIP, DIP and AP

For superintuitionistic logics:

- CIP $\Leftrightarrow$ DIP   by the deduction theorem,
- CIP $\Rightarrow$ Beth definability property (BDP)   by an easy calculation,
- in fact, BDP holds for any superintuitionistic logic (G. Kreisel),
- on the other hand, only 8 logics have the CIP (L.L. Maksimova).

Maksimova's results

To obtain her celebrated result, Makisimova used the following for superintuitionistic logics.

- CIP $\Leftrightarrow$ superAP $\Leftrightarrow$ AP.

Here, the superAP means AP with

for any $b \in B$ and any $c \in C$, if $h(b) \leq k(c)$ then there exists $a \in A$ such that $h(b) \leq hf(a) = kg(a) \leq k(c)$.

Also, the strongAP is defined to be the AP with

$$h(B) \cap k(C) = hf(A).$$

In general,

- superAP $\Rightarrow$ strongAP $\Rightarrow$ AP.

Robinson property

A substructural logic $\mathbf{L}$ has the Robinson property (RP), if the following holds:

Let X, Y and Z are sets of variables such that $X = Y \cap Z$, and let $Var(\Gamma) \subseteq Y$ and $Var(\Sigma) \subseteq Z$. Moreover, suppose that for each α such that $Var(\alpha) \subseteq X$

- $\Gamma \vdash_{\mathbf{L}} \alpha$ iff $\Sigma \vdash_{\mathbf{L}} \alpha$.

Then, for any formula ψ such that $Var(\psi) \subseteq Z$,

- $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi$ implies $\Sigma \vdash_{\mathbf{L}} \psi$.

How these properties are related?

RP for a logic $\mathbf{L}$ is equivalent to AP for the corresponding variety $V(\mathbf{L})$. (H.O. 1986)

For logics over $\mathbf{FL_e}$ (in fact, by local deduction theorem);

- CIP implies DIP,
- DIP is equivalent to RP,
- Hence, CIP for a logic $\mathbf{L}$ implies AP for $V(\mathbf{L})$.

IPs for logics over $\mathbf{FL_e}$

- By the local deduction theorem for $\mathbf{FL_e}$

SCIP	$\Leftrightarrow$	sup. RP	$\Leftrightarrow$	CIP

$$\Downarrow$$

SDIP	$\Leftrightarrow$	RP	$\Leftrightarrow$	DIP

IPs for logics over FL

- Parameterized local deduction theorem for **FL** is not strong enough to derive the DIP from the CIP. In fact,

$$
\begin{array}{ccccc}
\text{SCIP} & \Rightarrow & \text{sup. RP} & \Rightarrow & \text{CIP} \\
 & & \Downarrow & & \\
\Downarrow & & \text{str. RP} & & \\
 & & \Downarrow & & \\
\text{SDIP} & \Rightarrow & \text{RP} & \Rightarrow & \text{DIP}
\end{array}
$$

Algebraic characterizations

SCIP	$\Longrightarrow$	sup. RP $\Updownarrow$ sup. AP	$\Longrightarrow$	CIP
		$\Downarrow$		
$\Downarrow$		str. RP $\Updownarrow$ str. AP		
		$\Downarrow$		
SDIP	$\Longrightarrow$	RP $\Updownarrow$ AP	$\Longrightarrow$	DIP

Further algebraic characterizations

SCIP	$\Rightarrow$	sup. RP $\Updownarrow$ sup. AP	$\Rightarrow$	sup. IGAP	$\Rightarrow$	CIP $\Updownarrow$ sup. GAP
		$\Downarrow$		$\Downarrow$		$\Downarrow$
$\Downarrow$		str. RP $\Updownarrow$ str. AP	$\Rightarrow$	str. IGAP	$\Rightarrow$	str. GAP
		$\Downarrow$		$\Downarrow$		$\Downarrow$
SDIP	$\Rightarrow$	RP $\Updownarrow$ AP	$\Rightarrow$	DIP $\Updownarrow$ IGAP	$\Rightarrow$	GAP

Further development of Algebraic Logic

- We have shown that there are be close connections between *proof-theoretic* methods and *algebraic* methods.

- These connections will be shown not only for substructural logics but also for wider class of logics. Thus, algebraic logic can offer a suitable framework for discussing them.

- Much closer linkages between *logic* and *algebra* will be discovered in future, which surely lead us deeper understanding of both logic and algebra.

Appendices

Nuclei

A commutative monoid $\mathbf{M} = \langle M, \cdot, 1 \rangle$ is given. A unary function C on $\wp(M)$ is a nucleus if for all $X, Y \in \wp(M)$;

1. $X \subseteq C(X)$,
2. $CC(X) \subseteq C(X)$,
3. $X \subseteq Y$ implies $C(X) \subseteq C(Y)$,
4. $C(X) * C(Y) \subseteq C(X * Y)$.

Here, $W * Z = \{wz : w \in W, z \in Z\}$.

Nuclei as closure operators

Let C be a nucleus on $\wp(M)$. A subset X of M is C-closed if $C(X) = X$. Let $C(\wp(M))$ be the set of all C-closed subsets. Define operations $\cup_C$, $*_C$ and $\Rightarrow$ on $C(\wp(M))$ as follows: For all C-closed sets X and Y,

- $X \cup_C Y = C(X \cup Y)$,
- $X *_C Y = C(X * Y)$,
- $X \Rightarrow Y = \{z : X * \{z\} \subseteq Y\}$.

Then, we have the following:

> $\mathbf{C_M} = \langle C(\wp(M)); \cap, \cup_C, *_C, \Rightarrow, C(\{1\}), C(\emptyset)\rangle$ *is an* $\mathbf{FL_e}$*-algebra, not necessarily integral.*

Details of quasi-completions

Let $\mathbf{B}$ be a Gentzen matrix. For $x \in B^*$ and $a \in B \cup \{\epsilon\}$, define

$$[x; a] = \{w \in B^* : xw \preceq a\}.$$

Define an operation C on $\wp(B^*)$ by :

$C(X) = \cap\{[x; a] : X \subseteq [x; a]$ for $x \in B^*$ and $a \in B \cup \{\epsilon\}$ }.

C is a nucleus such that $C(\{\epsilon\}) = B^*$. Thus, $\mathbf{C_{B^*}}$ is a complete $\mathbf{FL_{ew}}$-algebra, which is called the quasi-completion of the $\mathbf{B}$.

Quasi-embeddings

To show that the Gentzen structure $\mathbf{B}$ is quasi-embeddable into $\mathbf{C}_{\mathbf{B}^*}$, define a map $k : B \to C(\wp(B^*))$, called a quasi-embedding, by

$$k(a) = [\epsilon; a] = \{w \in B^* : w \preceq a\}.$$

Then we have the following.

Suppose that $a, b \in B$ and that U and V are arbitrary C-closed subsets of B^* such that $a \in U \subseteq k(a)$ and $b \in V \subseteq k(b)$. Then for each $\star \in \{\wedge, \vee, \cdot, \to\}$,

$$a \star b \in U \star_C V \subseteq k(a \star b),$$

where $\star_C$ denotes $\cap, \cup_C, *_C$ and $\Rightarrow$, respectively.

Proof of cut elimination — concluded

Suppose that $g(s) \preceq g(t)$ is not true in $\mathbf{B}$ by a valuation g. Define a valuation h on $\mathbf{C}_{\mathbf{B}^*}$ by $h(q) = k(g(q))$ for each variable q. Then, by using induction on the length of the term r, we can show the following.

$$g(r) \in h(r) \subseteq k(g(r)) \text{ for each term } r.$$

Now, suppose contrary that $s \leq t$ holds in $\mathbf{C}_{\mathbf{B}^*}$. Then $h(s) \subseteq h(t)$. Using the above,

$$g(s) \in h(s) \subseteq h(t) \subseteq k(g(t)) = \{w : w \preceq g(t)\}.$$

But this implies $g(s) \preceq g(t)$, which is a contradiction. Thus, $s \leq t$ is not valid in $\mathbf{C}_{\mathbf{B}^*}$.

Beth definability property

Beth definability property (in the deductive form) (BDP)

A logic $\mathbf{L}$ has the Beth definability property, if for any formula $\alpha(\bar{x}_{m+1})$ and for all distinct variables y and z which are not members of $\bar{x}_{m+1}$,

if $\alpha(\bar{x}_m, y), \alpha(\bar{x}_m, z) \vdash_{\mathbf{L}} y \equiv z$ then there exists a formula $\delta(\bar{x}_m)$ such that

- $\alpha(\bar{x}_m, y) \vdash_{\mathbf{L}} y \equiv \delta(\bar{x}_m)$.

Strong DIP

This notion suggests us a strong form of the DIP, called the SDIP.

A logic $\mathbf{L}$ has the strong deductive interpolation property (SDIP), if for any set of formulas $\Gamma \cup \Sigma \cup \{\psi\}$, if $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi$ holds then there exists some δ such that

- $\Gamma \vdash_{\mathbf{L}} \delta$ and $\delta, \Sigma \vdash_{\mathbf{L}} \psi$,
- $Var(\delta) \subseteq Var(\Gamma) \cap Var(\Sigma \cup \{\psi\})$.

Equational IP

Another notion is the eqIP, an IP for *equational consequences*, introduced by A. Wroński.

A variety $\mathbf{V}$ of FL-algebras has the equational interpolation property (eqIP), if for any set of equations $G \cup E \cup \{\varepsilon\}$, if $G, E \models_{\mathbf{V}} \varepsilon$ holds then there exists a set of equations D such that

- $G \models_{\mathbf{V}} \delta$ for all $\delta \in D$, and $D, E \models_{\mathbf{V}} \varepsilon$,
- $Var(D) \subseteq Var(G) \cap Var(E \cup \{\varepsilon\})$.

Algebraization

The algebraization theorem for $\mathbf{FL}$ enables us to transfer results on the deduciblity relation $\vdash_{\mathbf{L}}$ of a substructural logic $\mathbf{L}$ into those on the equational consequence $\models_{\mathbf{V}}$ of the corresponding variety $\mathbf{V}$, and vice versa.

In particular:

> *A substructural logic* $\mathbf{L}$ *has the SDIP iff the variety* $\mathbf{V}(\mathbf{L})$ *has the eqIP*.