Algebraic Study of Lattice-Valued Logic and Lattice-Valued Modal Logic

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Outline

1. Introduction

2. On Algebras of Lattice-Valued Logic *L*-**VL**
   - Lattice-valued semantics
   - Algebraic axiomatization of *L*-**VL**
   - Prime *L*-filters and a Stone-type representation

3. On Algebras of Lattice-Valued Modal Logic *L*-**ML**
   - Lattice-valued Kripke semantics
   - Algebraic axiomatization of *L*-**ML**
   - A Jónsson-Tarski-type representation

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Historical Background

- In 1991 Fitting introduced L-valued modal logics for a finite distributive lattice L, which are endowed with all truth constants corresponding to the elements of L.
- He developed sequent calculi and tableau methods.
- Koutras and others studied model theoretic properties.
- But there seems to be no algebraic semantics.
- We develop algebraic semantics and representation theory for a modified Fitting's L-valued modal logic.
- We also show the finite model property and a Gödel-Tarski-McKinsey-type theorem between L-val. intuitionistic logic and L-val. modal logic of **S4** type.

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Truth Constants vs. T_a's

We consider Fitting's L-valued modal logic modified by replacing truth constants with unary connectivs T_a's for $a \in L$.

- $\mathrm{T}_a(x)$ intuitively states: The truth value of x is a.
- Note: In our setting, a truth constant $a \in L$ can be represented by adding the axiom $\mathrm{T}_a(p)$.

This offers a technical advantage and a philosophical one. The technical advantage is as follows.

- Koutras and Eleftheriou mention the difficulty of developing algebraic semantics for Fitting L-valued logic in the paper:
  - "Frame constructions, truth invariance and validity preservation in many-valued modal logic", 2005
- By the modification, we can develop algebraic semantics and representation theory.

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- The existence of a fuzzy truth constant a ($\neq 0, 1$) philosophically means: There is a proposition such that the truth value of it is "always exactly" a.
- This seems to contradict our intuition, since:
- The truth value of a fuzzy proposition may vary from one possible world, one person or one time to another.
- In contrast T_a's do not have such ontological commitment.

There may be a expression having similar meaning to $\mathrm{T}_a(x)$ in our natural languages, though there may be no expression having similar meaning to a fuzzy truth constant.

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The algebra of truth values

- L denotes a finite distributive lattice.
  Thus L forms a Heyting algebra.
- **2** denotes the two-element Boolean algebra.

Definition

We endow L with the unary operations $\mathrm{T}_a(\text{-})$'s for all $a \in L$ defined by, for $x \in L$,

$$\mathrm{T}_a(x) = \begin{cases} 1 & (\text{if } x = a) \\ 0 & (\text{if } x \neq a). \end{cases}$$

$\mathrm{T}_a(x)$ intuitively states: The truth value of x is a.
N/B: If $L = \mathbf{2}$ then $\mathrm{T}_1(x) = x$ and $\mathrm{T}_0(x) = \neg x$.

L-valued semantics

We define L-valued logic L-**VL** by L-valued semantics.

- The connectives of L-**VL**: $\wedge$, $\vee$, $\rightarrow$, 0, 1, T_a (for all $a \in L$).
- **Form** denotes the set of formulas of L-**VL**.

Definition

v is an L-valuation iff
v is a function from **Form** to L and satisfies:

$$\begin{aligned} v(\mathrm{T}_a(x)) &= \mathrm{T}_a(v(x)); \\ v(x@y) &= v(x)@v(y) \ \text{ for } \ @ = \wedge, \vee, \rightarrow; \\ v(a) &= a \ \text{ for } \ a = 0, 1. \end{aligned}$$

x is a valid formula of L-**VL** iff $v(x) = 1$ for all L-valuations v.

L-**VL**-algebras

We give an algebraic axiomatization of *L*-**VL** with completenss.
$x \leq y$ denotes $x \wedge y = x$. $x \leftrightarrow y$ denotes $(x \to y) \wedge (y \to x)$.

Definition (*L*-VL-algebras, main result)

$(A, \wedge, \vee, \to, \mathrm{T}_a\ (a \in L), 0, 1)$ is an *L*-**VL**-algebra iff it satisfies:
(i) $(A, \wedge, \vee, \to, 0, 1)$ forms a Heyting algebra;
(ii) $\mathrm{T}_a(x) \wedge \mathrm{T}_b(y) \leq \mathrm{T}_{a \to b}(x \to y) \wedge \mathrm{T}_{a \wedge b}(x \wedge y) \wedge \mathrm{T}_{a \vee b}(x \vee y)$,
$\mathrm{T}_b(x) \leq \mathrm{T}_{\mathrm{T}_a(b)}(\mathrm{T}_a(x))$;
(iii) $\mathrm{T}_0(0) = 1$, $\mathrm{T}_a(0) = 0$ $(a \neq 0)$, $\mathrm{T}_1(1) = 1$, $\mathrm{T}_a(1) = 0$ $(a \neq 1)$;
(iv) $\mathrm{T}_1(\mathrm{T}_a(x)) = \mathrm{T}_a(x)$, $\mathrm{T}_0(\mathrm{T}_a(x)) = \mathrm{T}_a(x) \to 0$,
$\mathrm{T}_b(\mathrm{T}_a(x)) = 0$ (for $b \neq 0, 1$);
(v) $\mathrm{T}_1(x) \leq x$, $\mathrm{T}_1(x \wedge y) = \mathrm{T}_1(x) \wedge \mathrm{T}_1(y)$;
(vi) $\bigvee\{\mathrm{T}_a(x)\ ;\ a \in L\} = 1$, $\mathrm{T}_a(x) \wedge \mathrm{T}_b(x) = 0$ (for $a \neq b$);
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Prime L-filters

A denotes an L-**VL**-algebra in this section.

Definition (L-filters)

An L-filter of A is a non-empty proper upper subset of A which is closed under $\wedge$ and T_1.

Definition (Prime L-filters, main result)

A prime L-filter of A is an L-filter of A such that

$$\mathrm{T}_c(x \vee y) \in P \Rightarrow \exists a, b \in L \ (a \vee b = c \text{ and } \mathrm{T}_a(x), \mathrm{T}_b(y) \in P).$$

If L is totally ordered, then we have:

- An L-filter P is prime iff $x \vee y \in P \Rightarrow x \in P$ or $y \in P$.

But, in the general case, this usual def. of primeness does not work well, e.g., for Stone-type rep. Thus we use the above def.

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Prime L-filter Theorem

We can prove the following “prime L-filter theorem."

Theorem

Let $x \neq y$ for $x, y \in A$. Then, there are $a \in L$ and a prime L-filter P of A such that $\mathrm{T}_a(x) \in P$ and $\mathrm{T}_a(y) \notin P$.

Definition

(i) P is an ultra L-filter of A iff $\forall x \in A\, \exists a \in L\, \mathrm{T}_a(x) \in P$.
(ii) P is a maximal L-filter iff P is maximal by inclusion.

If $L = \mathbf{2}$, then ultra L-filters coincide with ultrafilters.

Lemma

Prime L-filters, ultra L-filters and maximal L-filters all coincide.

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Stone-type representation of L-VL-algebras

For a set S, L^S denotes the set of all functions from S to L. L^S is endowed with the operations defined pointwise.

- A Boolean algebra is embedded into a powerset algebra.
- We show a more general result: An L-**VL**-algebra is embedded into an "L-valued powerset algebra" L^S.

$\mathrm{Spec}_L(A)$ denotes the set of all prime L-filters of A.

Definition (v_P)

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By the prime *L*-filter theorem, we obtain:

Theorem (Stone-type representation, main result)

Let $S = \mathrm{Spec}_L(A)$. *Define* $\Phi : A \to L^S$ *by*

$$\Phi(x) = (v_P(x))_{P \in S}.$$

Then, Φ *is an embedding, i.e., an injective homomorphism.*

By the above theorem, we have:

- The class of *L*-**VL**-algebras coincides with $\mathbb{ISP}(L)$.

Thus, since L forms a semi-primal algebra,
it follows from Natural Duality Theory by Davey and Clark that:

- A natural duality holds for *L*-**VL**-algebras.

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We define unary connectives U_a for $a \in L$:

Definition

For $a \in L$, define $\mathrm{U}_a(x) = \bigvee\{\mathrm{T}_b(x)\,;\ a \leq b\}$.

Lemma

For $x \in L$, $\mathrm{U}_a(x) = 1$ *(if* $a \leq x$*) and* $\mathrm{U}_a(x) = 0$ *(otherwise)*

Thus $\mathrm{U}_a(x)$ intuitively states:
The truth value of x is more than or equal to a.

Lemma

$\mathrm{T}_a(x) = \mathrm{U}_a(x) \wedge (\bigwedge\{\mathrm{U}_b(x) \to 0\,;\ a < b\})$ *holds in any L-VL-alg.*

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Jónsson-Tarski-type representation of *L*-**VL**-algebras

A referee of our paper pointed out:

- U_a's can be seen as boxes, i.e., $\mathrm{U}_a(x \wedge y) = \mathrm{U}_a(x) \wedge \mathrm{U}_a(y)$.
- Thus we can develop a J-T-type rep. of *L*-**VL**-alg. by a result in "A Sahlqvist Theorem for Distributive Modal Logic" (Gehrke, Nagahashi, Venema; APAL 2005)

The Priestley dual of the lattice reduct A^* of A is endowed with relations corresponding to each U_a. Using the relations, the lattice $\mathcal{D}$ of all down-sets of it is endowed with boxes which coincides with U_a's on the image of the embedding $A^* \hookrightarrow \mathcal{D}$.

- The Stone-type rep. is based on prime *L*-filters and requires no relation on dual spaces.
- The J-T-type rep. is based on prime filters and requires some relations on dual spaces.

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L-valued Kripke semantics

We define L-val. modal logic L-**ML** by L-val. Kripke semantics.

- The connectives of L-**ML**:
  a unary connective $\Box$ plus the connectives of L-**VL**.
- $\mathbf{Form}_{\Box}$ denotes the set of formulas of L-**ML**.

Definition

Let (M, R) be a Kripke frame. Then v is a Kripke L-valuation on (M, R) iff v is a function from $M \times \mathbf{Form}_{\Box}$ to L and satisfies:

$$\begin{aligned} v(w, \Box x) &= \bigwedge \{v(w', x)\,;\ wRw'\}; \\ v(w, \mathrm{T}_a(x)) &= \mathrm{T}_a(v(w, x)); \\ v(w, x@y) &= v(w, x)@v(w, y) \ \text{ for } @ = \wedge, \vee, \rightarrow; \\ v(w, a) &= a \quad \text{for } a = 0, 1. \end{aligned}$$

(M, R, v) is called an L-valued Kripke model.

Gödel translation from *L*-**IL** to *L*-**S4**

- *L*-val. int. logic *L*-**IL** is defined by *L*-val. Kripke semantics: $v(w, x \to y) = \bigwedge\{v(w', x) \to v(w', y)\ ;\ wRw'\}$; the remaining parts are the same as *L*-**ML**.
- *L*-val. modal logic of **S4** type *L*-**S4** is defined by letting *R* be reflexive and transitive.
- Gödel translation *G* from *L*-**IL** to *L*-**S4** is defined as follows: $G(p) = \Box p$ for prop. var. p; $G(x \to y) = \Box(G(x) \to G(y))$; $G(\mathrm{T}_a(x)) = \mathrm{T}_a(G(x))$; $G(x?y) = G(x)?G(y)$ for $? = \wedge, \vee$.

Theorem (Gödel-Tarski-McKinsey-type theorem)

The following are equivalent:

- *x is a valid formula of L-**IL**;*
- *$G(x)$ is a valid formula of L-**S4**.*

Gödel translation from L-IL to L-S4

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L-**ML**-algebras

We give an algebraic axiomatization of L-**ML** with completenss.

Definition (L-ML-algebras, main result)

$(A, \wedge, \vee, \rightarrow, \mathrm{T}_a\ (a \in L), \Box, 0, 1)$ is an L-**ML**-algebra iff it satisfies:

- $(A, \wedge, \vee, \rightarrow, \mathrm{T}_a\ (a \in L), 0, 1)$ forms an L-**VL**-algebra;
- $\Box(x \wedge y) = \Box x \wedge \Box y$, $\Box 1 = 1$;
- $\Box \mathrm{U}_a(x) = \mathrm{U}_a(\Box x)$ for all $a \in L$.

By an L-valued version of the filtration method, we can show:

Theorem (Finite Model Property)

The following are equivalent:

- $x = y$ *holds in any* L-**ML**-*algebras;*
- $v(w, x) = v(w, y)$ *in any finite* L*-valued Kripke models.*

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L-canonical models

Let A be an L-**ML**-algebra.
Recall: $v_p : A \to L$ is defined by $v_p(x) = a \Leftrightarrow \mathrm{T}_a(x) \in p$.

Definition (L-canonical models)

For $p \in \mathrm{Spec}_L(A)$ and $a \in L$, let

$$p_a = \{\mathrm{U}_a(x)\ ;\ \mathrm{U}_a(\Box x) \in p\}.$$

Define a binary relation $R_\Box$ on $\mathrm{Spec}_L(A)$ by

$$pR_\Box q \Leftrightarrow \forall a \in L\ p_a \subset q.$$

Define $v : \mathrm{Spec}_L(A) \times \mathbf{Form} \to L$ by $v(p, x) = v_p(x)$. Then $(\mathrm{Spec}_L(A), R_\Box, v)$ is called the L-canonical model of A.

The completeness is shown by using the L-canonical model.

Jónsson-Tarski-type representation of L-**ML**-algebras

Let A be an L-**ML**-algebra and $S = \mathrm{Spec}_L(A)$.
$\Box$ on L^S is defined as follows.

Definition ($\Box$ on L^S)

For $f \in L^S$, define $\Box f \in L^S$ by

$$(\Box f)(p) = \bigwedge \{f(q) \,;\, pR_{\Box}q\}.$$

Recall: $\Phi : A \to L^S$ is defined by $\Phi(x) = (v_P(x))_{P \in S}$.

Theorem (Jónsson-Tarski-type representation, main result)

$\Phi : A \to L^S$ preserves $\Box$, i.e., $\Phi(\Box x) = \Box\Phi(x)$.
*Hence $\Phi : A \to L^S$ is an embedding between L-**ML**-algebras.*

By letting $L = \mathbf{2}$, we can recover Jónsson-Tarski representation of modal algebras ($=$ **2**-**ML**-algebras).

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Conclusions and Future Work

In this work we obtained:

- Algebraic semantics for L-val. logic and L-val. modal logic.
- A Stone-type representation of L-**VL**-algebras using the notion of prime L-filters.
- A Jónsson-Tarski-type representation of L-**ML**-algebras.
- The finite model property of L-valued modal logic.
- A Gödel-Tarski-McKinsey-type theorem between L-val. intuitionistic logic and L-val. modal logic of **S4** type.

Our future work will be:

- Extend the representation of L-**ML**-algs. to a full duality based on the natural duality for L-**VL**-algebras.

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Thank you for your attention!!