Graded Consequence Relation and its Level-Cuts

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Inter-relation of the object level negation with the notion of consequence determines the notion of inconsistency of a logical system.

Inconsistency: Classical standpoint

There are two usages of the notion of inconsistency or contradictory set.

1. Negation-inconsistency: A set of formulae is inconsistent if a formula and its negation both follow from it.

2. Explosive usage or Absolute inconsistency: A set is considered to be a contradictory set if it allows any formula to follow.

Paraconsistent standpoint to understand an inconsistent set does not accept the explosive usage.

Paraconsistent Logic: A logic is paraconsistent if its notion of consequence is non-explosive i.e. it is not that from $\{\alpha, \neg\alpha\}$ any formula follows.

Why Paraconsistent Logic?

"Commitment to a contradiction does not seem to compel rationally (or even to make rationally more plausible) commitment to absolutely everything whatsoever."

_ Greg Restall

Concisely, this is the reason behind the emergence of paraconsistent logics and more mathematically, paraconsistent logic was initiated in order to challenge the spread law that anything follows from a contradictory set.

“There is widespread acknowledgement that the law of non-contradiction is an important logical principle. However there is less than universal agreement on exactly what the law amounts to. This unclarity was brought to light by the emergence of paraconsistent logics in which contradictions are tolerated; from the point of view of proofs, not everything need to follow from a contradiction and from the point of view of models, there are ‘worlds’ in which contradictions are true.”

-Greg Restall

The first approach leads to paraconsistency and the second approach leads to dialetheism.

‘Paraconsistency’ and ‘Dialetheism’ are not synonyms, rather any rational version of the latter needs the former but the converse seems not to hold. But the point where both the versions overlap is to negate the spread law.

Classical Notion of Consequence

Classical notion of consequence is a function C: P(F) $\rightarrow$ P(F) , where F is the set of wffs, and P(F) , the power set of F, mapping each set of formulae X to its consequence set C(X) satisfying,

C1. $X \subseteq C(X)$

C2. If $X \subseteq Y$ then $C(X) \subseteq C(Y)$

C3. $C(C(X)) = C(X)$

C4. $C(X) = \cup_{Y \subseteq X,\ Y \text{ is finite}} C(Y)$

C5. $\alpha \supset \beta \in C(X)$ iff $\beta \in C(X \cup \{\alpha\})$

C6. $C(\{\alpha, \neg\alpha\}) = F$

C7. $C(\{\alpha\}) \cap C(\{\neg\alpha\}) = C(\Phi)$

Classical Notion of inconsistency

INCONS is a unary relation over P(F) satisfying,

1. If $X \subseteq Y$ then $X \in$ INCONS implies $Y \in$ INCONS
2. If $X \in$ INCONS then there is a finite subset Y of X such that $Y \in$ INCONS
3. If $X \cup \{\alpha\} \in$ INCONS and $X \cup \{\neg\alpha\} \in$ INCONS then

   $X \in$ INCONS
4. $\{\alpha, \neg\alpha\} \in$ INCONS
5. $X \cup \{\alpha, \neg\beta\} \in$ INCONS iff $X \cup \{\neg(\alpha \supset \beta)\} \in$ INCONS

Equivalence between classical consequence and inconsistency

Theorems for conversion:

1. Let C be a classical consequence operator, postulated by the axioms C1- C7. Let INCONS be defined by X $\in$ INCONS iff for any α,$\{\alpha, \neg\alpha\} \subseteq C(X)$.then INCONS satisfies all the classical inconsistency axioms.

2. Let INCONS be a unary relation satisfying classical inconsistency axioms, Incons1– Incons5.Let C be defined by,

   $\alpha \in C(X)$ iff $X \cup \{\neg\alpha\} \in$ INCONS . Then C satisfies all classical consequence axioms.

   It is also established that if we start from C, obtain the INCONS by (1) and then from INCONS obtain C' by (2) then

   $C = C'$. Similar in the case for other way round.

Truncated version of classical consequence axioms assuming only the presence of negation in the language

Let C: P(F) → P(F) be a mapping, where F is the set of wffs, satisfying,

(C1′) $X \subseteq C(X)$

(C2′) If $X \subseteq Y$ then $C(X) \subseteq C(Y)$

(C3′) $C(C(X)) = C(X)$

(C4′) $C(X \cup \{\alpha\}) \cap C(X \cup \{\neg\alpha\}) = C(X)$

(C5′) $C(\{\alpha, \neg\alpha\}) = F$

Corresponding notion of Inconsistency

Let INCONS be a unary relation over P(F) satisfying

Incons1. If $X \subseteq Y$ then $X \in$ INCONS implies $Y \in$ INCONS

Incons2. If $X \cup \{\neg\alpha\} \in$ INCONS, for each α in Y and $X \cup Y \in$ INCONS then $X \in$ INCONS

Incons3. $\{\alpha, \neg\alpha\} \in$ INCONS

Graded Consequence Relation

A graded consequence relation is a fuzzy relation |~ from *P(F)* to *F*, satisfying,

1. If $\alpha \in X$ then gr(X|~ α) = 1 (Overlap)
2. If $X \subseteq Y$ then gr(X|~ α) ≤ gr(Y|~ α) (Monotonicity)
3. $\inf_{\beta \in Z}$ gr(X|~β) * gr($X \cup Z$|~ α) ≤ gr(X|~ α) (Cut)

where 'inf' and '*' of a complete Residuated lattice, are used to compute meta linguistic 'for all' and 'and' respectively.

We consider a complete pseudo Boolean algebra <L, →, ∧, v, 0,1> as the meta level algebraic structure.

Extension of Graded Consequence Relation in presence of object level negation (¬)

GC1. If $\alpha \in X$ then $gr(X|\sim \alpha) = 1$

GC2. If $X \subseteq Y$ then $gr(X|\sim \alpha) \leq gr(Y|\sim \alpha)$

GC3. $\inf_{\beta \in Z} gr(X|\sim \beta) \wedge gr(X \cup Z|\sim \alpha) \leq gr(X|\sim \alpha)$

GC4. There is some $k>0$ such that for any α,

$\inf_{\beta} gr(\{\alpha, \neg\alpha\} |\sim \beta) \geq k$

GC5. $gr(X \cup \{\alpha\}|\sim \beta) \wedge gr(X \cup \{\neg\alpha\}|\sim \beta) \leq gr(X|\sim \beta)$

Graded Inconsistency

Let INCONS be a fuzzy subset over the set of all sets of formulae. For each set of formulae X, INCONS(X), the degree to which X is inconsistent is postulated by,

1. If $X \subseteq Y$ then $INCONS(X) \leq INCONS(Y)$
2. $INCONS(X \cup \{\neg y\}) \wedge INCONS(X \cup Y) \leq INCONS(X)$, for each y in Y
3. There is some $k>0$ such that for any α,

   $INCONS(\{\alpha, \neg \alpha\}) \geq k$

Equivalence between graded consequence and graded inconsistency

As in classical case, graded notion of consequence and inconsistency both are equivalent i.e.

1. Given a graded consequence relation |~, graded notion of inconsistency can be obtained by defining INCONS by, $INCONS(X)=\inf_{\alpha} gr(X|\sim \alpha)$
2. Given INCONS satisfying all graded inconsistency axioms, a graded consequence relation can be defined by

$$gr(X|\sim \alpha) = 1 \ , \ \text{if } \alpha \in X$$
$$= INCONS(X \cup \{\neg \alpha\}) \ , \text{otherwise}$$

Level-cuts of a graded consequence relation

Let |~ be a graded consequence relation. For each $i \in L$ let us define

$C_i(X) = \{ \beta / \ gr(X \mid\sim \beta) \geq i \}$

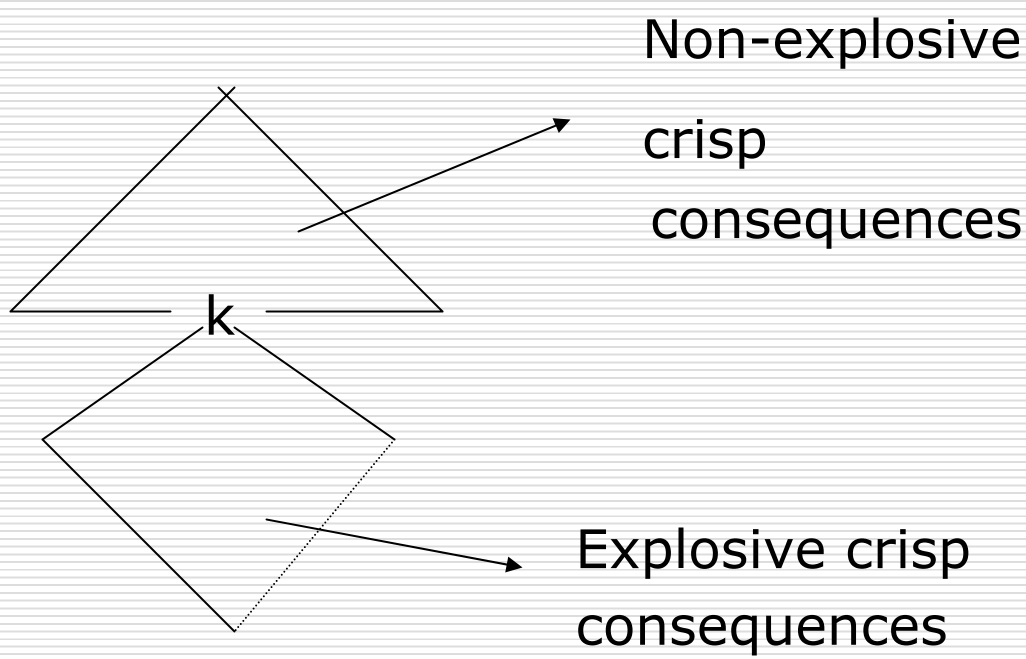
C_i turns out to be a Tarskian consequence operator in the sense that C_i satisfies

(i) $X \subseteq C_i (X)$

(ii) If $X \subseteq Y$ then $C_i (X) \subseteq C_i (Y)$

(iii) $C_i(C_i(X)) = C_i(X)$

Theorem: For each $i \in L$, $C_i(\{\alpha, \neg \alpha\}) = F$ for any α, if $i \leq k$ and there is some α, for which $C_i(\{\alpha, \neg \alpha\}) \neq F$ if $i > k$ or i is non-comparable to k.



Proof:

We shall prove the theorems in two stages.

Stage-1:

Let us take an arbitrary α1 such that

$\inf_\beta gr(\{\alpha_1 , \sim \alpha_1 \} |\!\sim \beta) = a \geq k$

Now for any i ∈ L, either $i \leq a$ or $i > a$ or i is non-comparable with *a*.

Case-I Let $i \leq a$

As $\inf_\beta gr(\{\alpha_1 , \sim \alpha_1 \} |\!\sim \beta) = a$, for all β,

$gr(\{\alpha_1 , \sim \alpha_1 \} |\!\sim \beta) \geq a \geq i$

$\therefore C_i(\{\alpha_1, \sim \alpha_1 \}) = F$

Case-II Let $i > a$

$$\begin{aligned} C_i(\{\alpha_1, \sim \alpha_1 \}) &= \{\beta \,/\, gr(\{\alpha_1, \sim \alpha_1 \} |\!\sim \beta) \geq i \} \\ &\subseteq \{\beta \,/\, gr(\{\alpha_1, \sim \alpha_1 \} |\!\sim \beta) \geq a \} \\ &= C_a(\{\alpha_1, \sim \alpha_1 \}) = F \end{aligned}$$

Claim is that $Ci(\{α_1, \sim α_1\})$ is a proper subset of $C\textit{a}(\{α_1, \sim α_1 \})$
i.e. there is some β which does not belong to $Ci(\{α_1, \sim α_1 \})$.
If not, then for all β, $gr(\{α_1, \sim α_1 \}|\sim β) \geq i$
$\therefore$ $\inf_β gr(\{α_1 , \sim α_1 \}|\sim β) = i > a$
This is a contradiction to the assumption $\inf_β gr(\{α1 , \sim α1 \}|\sim β) = a$
$\therefore$ $Ci(\{α_1, \sim α_1 \}) \neq F$

<u>Case-III</u> Let *a* and i be non-comparable and sup { *a* , i}= j
As *a* and i are non-comparable and $gr(\{α_1 , \sim α_1 \}|\sim β) \geq a$, for all β, for no γ, $gr(\{α_1 , \sim α_1 \}|\sim γ) = i$
$\therefore$ $Ci(\{α_1, \sim α_1 \}) = \{β / gr(\{α_1, \sim α_1 \}|\sim β) > i \}$
Claim is that $Ci(\{α_1, \sim α_1 \}) = Cj(\{α_1, \sim α_1\})$ i.e.
$\{β / gr(\{α_1, \sim α_1 \}|\sim β) > i \}= \{β / gr(\{α_1, \sim α_1 \}|\sim β) \geq j \}$ i.e.there is no such β such that $i < gr(\{α_1, \sim α_1 \}|\sim β) < j$ or $i < gr(\{α_1, \sim α_1 \}|\sim β)$ but $gr(\{α_1, \sim α_1 \}|\sim β)$ is non-comparable with j. --------(a)

For the first case, if possible let for some β,

$i < gr(\{\alpha_1, \sim \alpha_1\} | \sim \beta) = l < j$

$\therefore l \geq a$ [Since for all γ, $gr(\{\alpha_1, \sim \alpha_1\} | \sim \gamma) \geq a$]

But $i < l < j$ together with $l \geq a$ contradict the fact that $\sup \{a, i\} = j$

That is there is no such β such that $i < gr(\{\alpha_1, \sim \alpha_1\} | \sim \beta) < j$

For the second case, let us assume there is some β for which

$i < gr(\{\alpha_1, \sim \alpha_1\} | \sim \beta) = l$ but l and j are non-comparable.

Then again as $a \leq l$, l is an upper bound of *a* and i. Then as

$j = \sup \{a, i\}$, j can not be non-comparable with l.

Hence (a) is proved.

$\therefore Ci(\{\alpha_1, \sim \alpha_1\}) = Cj(\{\alpha_1, \sim \alpha_1\}) \neq F$ [By case-II, since $j > a$]

Case-II Let $i > k$

Then three subcases arise. Subcase (i) $a < i$ Subcase (ii) a and i are non comparable Subcase (iii) $k < i < a$

For (i) and (ii) as we already have in stage-1

$\inf\beta gr(\{\alpha_1, \sim \alpha_1\} |\!\sim \beta) = a$, by case-I and case-II of stage-1 we can conclude

$Ci(\{\alpha_1, \sim \alpha_1\}) \neq F$

Subcase (iii) Let $k < i < a$

Since $k < i$, it is not that for any α, $\inf\beta gr(\{\alpha, \sim \alpha\} |\!\sim \beta) \geq i$

i.e. there is some α_2 such that $\inf\beta gr(\{\alpha_2, \sim \alpha_2\} |\!\sim \beta) = j$ where either $j < I$ or j is non-comparable with i.

Then in either cases $Ci(\{\alpha_2, \sim \alpha_2\}) \neq F$ [By already proved results in stage-1]

Combining all these three cases, we can conclude that there is some α such that

$Ci(\{\alpha, \sim \alpha\}) \neq F$, for $i > k$

Case-III Let i be non-comparable with k

Then again two subcases arise.

Subcase (i) *a* and i are non-comparable

Subcase (ii) i ≤ *a*

[As k and i are non-comparable and k ≤ *a*, the case for *a* > i would not arise.]

For subcase (i) again as infβgr({α_1 , ~ α_1 }|~β) = *a*, by previous result Ci({α_1, ~ α_1 }) ≠ F

Subcase (ii) Let i ≤ *a*

Since i is non-comparable with k, there is no such γ such that infβgr({γ , ~ γ }|~β) = i

Also it is not that for any α, infβgr({α , ~ α }|~β) > i

$\therefore$ There is some ɑ3 such that infβgr($\{\alpha_3, \sim \alpha_3\} | \sim \beta$) = j, where j < i or j is non-comparable with i.

But j < i can not be the case. Because if j < i then as j $\geq$ k, we have k < i. This contradicts the assumption that i is non-comparable with k.

Now if j is non-comparable with i, Ci($\{\alpha_3, \sim \alpha_3\}$) $\neq$ F

[By previous result as infβgr($\{\alpha_3, \sim \alpha_3\} | \sim \beta$) = j]

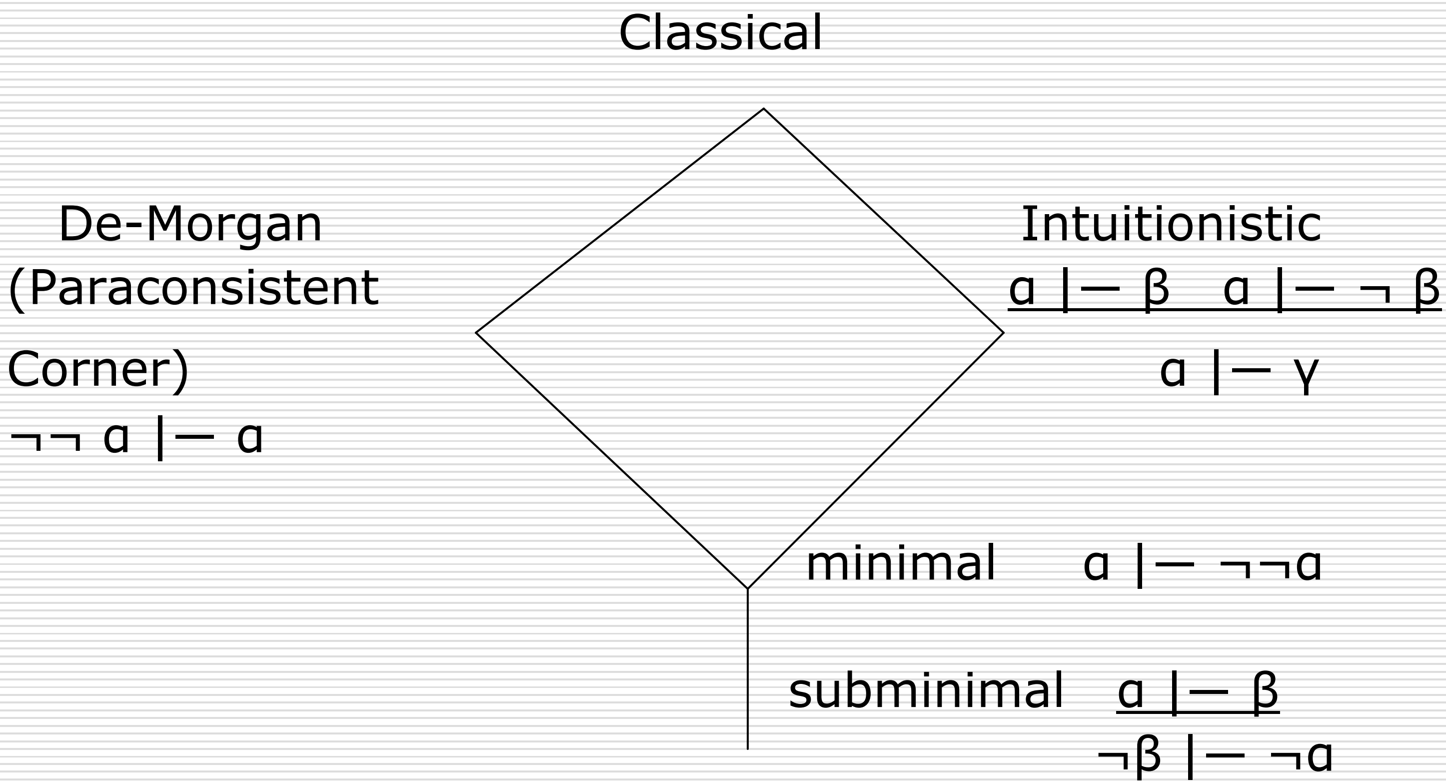
Combining all the above subcases, we can conclude that there is some ɑ such that

Ci($\{\alpha, \sim \alpha\}$) $\neq$ F, where i is non-comparable with k.

Profile of the next slides

1. In the context of graded consequence a survey is made by imposing different conditions on the object level negation.
2. Corresponding changes in the crisp consequences generated from a graded consequence relation.
3. A study of the notion of inconsistency relative to the crisp consequences generated from a graded consequence relation.

Diagram on Negation



Addition of GC6 and GC7

GC6. $gr(X \cup \{\alpha\} \mid\sim \beta) \leq gr(X \cup \{\neg\beta\} \mid\sim \neg\alpha)$

[conterpart of subminimal negation in the context of graded consequence]

GC7. $gr(X \mid\sim \alpha) \leq gr(X \mid\sim \neg\neg\alpha)$

[counterpart of minimal negation in the context of graded consequence]

A survey on the counterpart of De-Morgan negation in the context of graded consequence

$gr(X|\sim \neg\neg\alpha) \leq gr(X|\sim \alpha)$

\+ GC1, GC2, GC6

\+ GC1, GC2, GC5, GC6

$gr(X |\sim \alpha) \wedge gr(X|\sim \neg\alpha) \leq gr(X|\sim \beta)$

GC4 with k=1

Properties of crisp consequences generated from a graded consequence relation (GC1-GC7)

<u>For any i ≤ k of L</u>

- $X \subseteq C_i(X)$
- If $X \subseteq Y$ then $C_i(X) \subseteq C_i(Y)$
- $C_i C_i(X) = C_i(X)$
- $C_i(\{\alpha, \neg\alpha\}) = F$
- $C_i(X \cup \{\alpha\}) \cap C_i(X \cup \{\neg\alpha\}) = C_i(X)$
- $\beta \in C_i(X \cup \{\alpha\})$ implies $\neg\alpha \in C_i(X \cup \{\neg\beta\}$
- $\alpha \in C_i(X)$ implies $\neg\neg\alpha \in C_i(X)$

<u>Otherwise</u>

- $X \subseteq C_i(X)$
- If $X \subseteq Y$ then $C_i(X) \subseteq C_i(Y)$
- $C_i C_i(X) = C_i(X)$
- For some α, $C_i(\{\alpha, \neg\alpha\}) \neq F$
- $C_i(X \cup \{\alpha\}) \cap C_i(X \cup \{\neg\alpha\}) = C_i(X)$
- $\beta \in C_i(X \cup \{\alpha\})$ implies $\neg\alpha \in C_i(X \cup \{\neg \beta\}$
- $\alpha \in C_i(X)$ implies $\neg\neg\alpha \in C_i(X)$

Inconsistency generated by $C_{i,}$ i $\leq$k

INCONS, a binary relation over P(F), the set of all sets of formulae satisfying,

Incons1. If $X \subseteq Y$ then $X \in$ INCONS implies $Y \in$ INCONS

Incons2. If $X \cup \{\neg \alpha\} \in$ INCONS, for each α in Y and $X \cup Y \in$ INCONS then $X \in$ INCONS

Incons3. $\{\alpha, \neg \alpha\} \in$ INCONS

Incons4. $X \cup \{\alpha\} \in$ INCONS implies $X \cup \{\neg\neg \alpha\} \in$ INCONS

Inconsistency generated by C_i, $i>k$ or i is non-comparable to k

Non-explosive consequence generates a formula-dependent notion of inconsistency

A consequence is non-explosive means it is not that for all α, $C(\{\alpha, \sim\alpha\}) = F$ i.e. for some α, $C(\{\alpha, \sim\alpha\}) \neq F$ but there may exist some β for which

$C(\{\beta, \sim \beta\}) = F$

This gives an indication that a non-explosive consequence depends on the formula under consideration and hence gives rise to a formula-dependent notion of inconsistency.

Example

- Tweety is a bird. (α_1)
- Birds can fly. (α_2)
- Tweety is a Penguin. (α_3)
- Penguins cannot fly. (α_4)

Hence

- Tweety can fly. (β)
- Tweety cannot fly. ($\neg\beta$)

i.e. $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$yields both β and$\neg\beta$.

But does the common sense allow$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$to yield both α_2 and $\neg$ α_2?

So it seems that $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is inconsistent with respect to β but not with respect to α_2.

Theorem:

Let us consider C_i, where i>k or i is non-comparable to k and define a notion ParaINCONS by

(X, α)$\in$ ParaINCONS iff{α, ~α} $\subseteq C_i(X)$. Then ParaINCONS satisfy

(PI)1. If $\alpha \in X$ then $(X \cup \{\neg\alpha\}, \alpha) \in$ ParaINCONS

(PI)2. If $X \subseteq Y$ then $(X, \alpha) \in$ ParaINCONS implies $(Y, \alpha) \in$ ParaINCONS

(PI)3. If for all $\alpha \in Y$, $(X \cup \{\neg\alpha\}, \alpha) \in$ ParaINCONS then $(X \cup Y, \beta) \in$ ParaINCONS implies $(X, \beta) \in$ ParaINCONS

(PI)4. For some α, there is some β such that $(\{\alpha, \neg\alpha\}, \beta) \notin$ ParaINCONS

(PI)5. $(X \cup \{\alpha\}, \beta) \in$ ParaINCONS and $(X \cup \{\neg\alpha\}, \beta) \in$ ParaINCONS imply $(X, \beta) \in$ ParaINCONS

(PI)6. $(X, \alpha) \in$ ParaINCONS iff $(X, \neg\alpha) \in$ ParaINCONS

(PI)7. $(X \cup \{\alpha\}, \beta) \in$ ParaINCONS iff $(X \cup \{\neg\neg\alpha)\}, \neg\neg\beta) \in$ ParaINCONS

Summary

- In the context of graded consequence, a survey is made by imposing standard conditions on negation – any reflexive, monotonic consequence is explosive if its object level negation obeys the law of contraposition and $X|\!-\neg\neg\alpha$ implies $X|\!-\alpha$.
- A graded consequence relation characterized by GC1 to GC7 gives rise to two sets of crisp consequences: one is explosive and the other is non-explosive.
- The set of explosive consequences generates a notion of inconsistency which is close to classical; the other gives rise to parainconsistency, the notion of inconsistency relative to a formula.

Thank you