

Instantial Relevance in Inductive Logic

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Joint work with Jeff Paris and Juergen Landes

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- Why this should be and also the problem of quantifying this effect has long been studied by Philosophers under the name of 'Induction'.

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- One particular approach developed by W. E. Johnson in the 1920's and R. Carnap et al in the 1940-50s involved studying it within the formal framework of 1st order logic and as such became known as Inductive Logic.
- They aimed to postulate rational (or logical) principles of belief formation and investigate their consequences for this and various related questions.
- The holy grail here would be to uncover such principles which were generally acceptable and uniquely determine, by purely logical considerations, how to quantify one's beliefs on the basis of one's knowledge.

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- **Framework** An agent inhabits a world in which there are just countably many *individuals* a_1, a_2, a_3, \dots , finitely many *properties or relations* R_1, R_2, \dots, R_k and no functions nor equality.
- The agent has no further knowledge of his environment and is required to assign probabilities $Pr(\theta)$ to sentences θ of this language (i.e. the language consisting of constants a_i and predicates R_j) 'rationally'.
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- In this context the above ‘general intuition’ might be captured as follows:
- Suppose our agent has ‘seen’ a_1, a_2, \dots, a_n in the sense of learning whether or not $R_j(a_{i_1}, a_{i_2}, \dots, a_{i_m})$ holds for each of the relations R_j and each $1 \leq i_1, i_2, \dots, i_m \leq n$. [Call the conjunction of such a set of literals a *state description*.]
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- Then the following version \mathcal{IR} of Instantial Relevance should hold:

If there are more amongst the a_1, \dots, a_n indistinguishable from a_i than there are indistinguishable from a_j then at least as much probability should be given to a_{n+1} also being indistinguishable from a_i as is given to it being indistinguishable from a_j .

- For example, if there is just one binary relation symbol R , a state description for four individuals may be represented by a 4×4 matrix with entries 0,1. Below is such a matrix.

$$\begin{array}{cccc} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{array}$$

Here, 1 or 0 at (i, j) th place means that $R(a_i, a_j)$ does or does not hold respectively. \mathcal{IR} states that the it is more probable that a_5 will be like a_1 (and a_3) then like either of a_2 and a_4 .

- What we would like are rational principles of probability assignment which force (and in a sense explain) this inequality.
- Of course we could simply postulate this as a principle. But that would be unsatisfactory as it would provide no form of *explanation*.

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Atom Exchangeability (Unary case) The probability of a state description depends only on the sizes of its indistinguishability classes.

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- The (shortest) proof uses two other classical theorems:
 - De Finetti's Representation Theorem, which implies that all probability functions satisfying Atom Exchangeability are mixtures of a family of some particularly simple probability functions satisfying Atom Exchangeability,
 - Muirhead's Theorem, which in its simplest form says that if we define for $0 < a, b \in \mathbb{R}$, $n, m \in \mathbb{N}$,

$$Q(m, n) = a^n b^m + a^m b^n$$

then $m < n$ implies

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- In this case the direct analog of Atom Exchangeability is called **Spectrum Exchangeability** (same definition, just different language)

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- Returning to our example with one binary relation symbol, the state descriptions represented by matrices

$$\begin{array}{cccc} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{array} \qquad \begin{array}{cccc} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{array}$$

should by Spectrum Exchangeability, get the same probability.

- To replicate the proof for the unary case requires to have, in the first place, a de Finetti style Representation Theorem for probability functions satisfying Spectrum Exchangeability
- Probability functions satisfying Spectrum Exchangeability need two types de Finetti style Representation Theorems. This contribution concerns one of them, valid for *Language Invariant* probability functions.

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- Probability functions satisfying Spectrum Exchangeability need two types de Finetti style Representation Theorems. This contribution concerns one of them, valid for *Language Invariant* probability functions.

- A Language Invariant family of probability functions is a *consistent* collection of probability functions Pr_L satisfying Spectrum Exchangeability, one for each finite set of predicates $L = \{R_1, \dots, R_k\}$. By 'consistent' we mean that the restriction of any Pr_L to a smaller set of predicates coincides with the appropriate member of the family.
- A probability function Pr is Language Invariant if it is a member of a Language Invariant family. Note that the above definition implies that Pr also satisfies Spectrum Exchangeability.

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- There is a de Finetti style Representation Theorem for Language Invariant probability functions: any such probability function is a mixture of some particularly simple Language Invariant probability functions.
- No time to explain the nature of these 'particularly simple' functions, but the paper shows that they satisfy \mathcal{IR} provided the following generalization of Muirhead Theorem holds:

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- No time to explain the nature of these 'particularly simple' functions, but the paper shows that they satisfy \mathcal{IR} provided the following generalization of Muirhead Theorem holds:

- Let X be a finite set of positive real numbers, $n, m \in \mathbb{N}$ and define

$$Q(m, n) = \sum_{\substack{S \cup T = X \\ S \cap T = \emptyset}} \left(\sum_{p \in S} p \right)^{\square m} \left(\sum_{q \in T} q \right)^{\square n}$$

where the \square in $\left(\sum_{p \in S} p \right)^{\square m}$ indicates that we take only those

terms from the expansion of $\left(\sum_{p \in S} p \right)^m$ which contain each element of S to a non-zero power etc., then $m < n$ implies

$$Q(m + 1, n) \leq Q(m, n + 1).$$

- After much effort, we have proved the generalization of Muirhead theorem, so in virtue of the representation theorem, all Language Invariant probability functions also satisfy \mathcal{IR} .

- An Example. Given (only) that

$$R(a_1, a_2), R(a_2, a_1), \neg R(a_2, a_3)$$

which should one rationally think the more probable, $R(a_3, a_2)$
or $\neg R(a_3, a_2)$?