2D Hybrid Logic of Spaces

Yi N. Wang

Philosophy Department
Peking University, Beijing, China
wonease@gmail.com

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A Somewhat Familiar Example

The real speed of a car: 121 kph
Policewoman’s radar gun: 121 kph
Accuracy of the radar gun: ±2 kph
Speed limit of the highway: 120 kph

Fact

- $(Speeding)$
- $\neg K_W (Speeding) \land \neg K_W \neg (Speeding)$
- The policewoman knows that she would have known whether the car was speeding if her radar gun had the accuracy of ±1 kph.
PROP: propositional variables.

The language:

\[ \phi ::= p \mid \neg \phi \mid \phi \land \psi \mid \lozenge \phi \mid \lozenge \lozenge \phi, \]

where \( p \in \text{PROP} \).

The semantics:

\[ \mathcal{S}, u, U \models p \iff u \in V(p) \]
\[ \mathcal{S}, u, U \models \lozenge \phi \iff \exists v \in U. \mathcal{S}, v, U \models \phi \]
\[ \mathcal{S}, u, U \models \lozenge \lozenge \phi \iff \exists V. (u \in V \subseteq U \& \mathcal{S}, u, V \models \phi), \]

where \( u \in U \) and \( \mathcal{S} = (S, \Sigma, V) \) is a subset model.
Truth of propositional variables relies only on points.

\[ u, U \models p \iff u \in V(p) \]
Graphs of the Semantics (II)

\[ u, U \models \diamond \varphi \]

\[ \mathcal{G}, u, U \models \diamond \varphi \quad \text{iff.} \quad \exists v \in U. \mathcal{G}, v, U \models \varphi \]

\[ \mathcal{G}, u, U \models \diamond \varphi \quad \text{iff.} \quad \exists X. (u \in X \subseteq U \& \mathcal{G}, u, X \models \varphi) \]
In the Sense of Epistemic Logic

- $\square$ can be taken as the ordinary $K$.
- $\Diamond$ operator shrinks the epistemic range, which refines the agent’s knowledge. This can be regarded as a sort of *epistemic effort* which is hard to be characterized by the classical epistemic logic.
Issues and Motivations

- Lack of scaling mechanism: the third fact in the previous example cannot be expressed;
- Adaptions to talk about belief (reflexivity should not hold):
  - Reinterpreting original modalities (e.g. using the derived set operation)
  - Adding new modalities (say, difference modality, cf. Kudinov [2])
Solutions or Alternative Ways

- Lack of scaling mechanism:
  *We add names for points and sets.*

- Adaptions to talk about belief, feelings and so on:
  - *We adopt neighborhood semantics and the neighborhood box operator to cover non-reflexive situations;*
  - *We use \(-\)-operator to express the “difference” modality.*
TWO-SORTED HYBRID LANGUAGES
AND
SPATIAL SEMANTICS
Naming Points and Neighborhoods

<table>
<thead>
<tr>
<th>New Atoms</th>
<th>NOM</th>
<th>SVAR</th>
</tr>
</thead>
<tbody>
<tr>
<td>PNT</td>
<td>PNTNOM</td>
<td>PNTVAR</td>
</tr>
<tr>
<td>SET</td>
<td>SETNOM</td>
<td>SETVAR</td>
</tr>
</tbody>
</table>

\[ \text{AT} = \text{PROP} \cup \text{NOM} \cup \text{SVAR} \]
Two-Sorted Hybrid Languages

Definition

The language $\mathcal{H}^2(@, \downarrow)$ is given by the following rule:

$$\varphi ::= \top | p | x | X | \neg \varphi | \varphi \land \psi | \lozenge \varphi | \lozenge \varphi | @_x \varphi | \downarrow^S \cdot \varphi$$

where $p \in \text{PROP}$, $x \in \text{PNT}$, $X \in \text{SET}$, $s \in \text{PNTVAR}$, $S \in \text{SETVAR}$. 
Hybrid subset model \((S, \Sigma, V)\):

- \(S\), a collection of *points*, is the domain;
- \(\Sigma \subseteq \varnothing S\) is a collection of *sets* (or *neighborhoods*);
- \(V : \text{PROP} \cup \text{NOM} \rightarrow S \cup \varnothing S\) is an evaluation mapping
  - every propositional variables to a set of points,
  - every point nominal to a point,
  - every set nominal to a set.

Two assignments \(g_0 : \text{PNTVAR} \rightarrow S\) and \(g_1 : \text{SETVAR} \rightarrow \Sigma\) are for the two sorts of nominals respectively.
Let $\mathcal{G} = (S, \Sigma, V)$ be a hybrid subset model, $g_0, g_1$ two assignments for points and sets. For every point $u$ and a neighborhood $U$ of $u$,

$\mathcal{G}, g_0, g_1, u, U \models x \iff u = x^{\mathcal{G}, g_0}$

$\mathcal{G}, g_0, g_1, u, U \models X \iff U = X^{\mathcal{G}, g_1}$

$\mathcal{G}, g_0, g_1, u, U \models @_x^X \varphi \iff \mathcal{G}, g_0, g_1, x^{\mathcal{G}, g_0}, X^{\mathcal{G}, g_1} \models \varphi$

$\mathcal{G}, g_0, g_1, u, U \models \downarrow_s^S \varphi \iff \mathcal{G}, g_0[u]^s, g_1[u]^s, u, U \models \varphi$

where $x \in \text{PNT}$, $X \in \text{SET}$, $s \in \text{PNTVAR}$, $S \in \text{SETVAR}$, and

$g_0[u]_s(s') = \begin{cases} u, & s = s', \\ g_0(s), & \text{otherwise.} \end{cases}$

The assignment $g_1$ is similar.
Graphs of the Semantics

\[ \sigma, g_0, g_1, u, U \models @^X_x \varphi \iff \sigma, g_0, g_1, x^{\sigma}, g_0, X^{\sigma}, g_1 \models \varphi \]

\[ \sigma, g_0, g_1, u, U \models \downarrow^S_s \varphi \iff \sigma, g_0[^u], g_1[^U], u, U \models \varphi \]
Axiomatization
The $\otimes$-prefixed Gentzen System $G_{H^2(\otimes,\downarrow)}$

Cf. pg. 201 in the Proceedings for the details.

- Cut is admissible;
- Quasi-subformula property;
- Soundness and completeness.
Internalizing the Semantics (Blackburn[1], Seligman[4])

We express the semantics in a two-sorted first-order language, and then internalize it into a hybrid logic.

\[ \mathcal{G}, g_0, g_1, u, U \models \Diamond \varphi \iff \exists V.(u \in V \subseteq U \land \mathcal{G}, g_0, g_1, u, V \models \varphi) \]

\[ R^{\Diamond \varphi} x X \leftrightarrow \exists Y.(x \in Y \subseteq X \land R^{\varphi} x Y) \]

\[ @^X_x \Diamond \varphi \leftrightarrow \exists Y.(x \in Y \subseteq X \land @^Y_x \varphi) \]
Going Further...
A structure $\mathcal{M} = (W, N, V)$ is called a hybrid neighborhood model, if the following hold:

- $W \neq \emptyset$;
- $N : W \rightarrow \wp(W)$, which is called a neighborhood function;
- $V : \text{PROP} \cup \text{NOM} \rightarrow W \cup \wp(W)$, where $V(p) \in \wp(W)$, $V(a) \in W$. 

Spatial Semantics (IV): A Unified Semantics

For subset models: define $Q u U :\Leftrightarrow u \in U$;

For neighborhood semantics: define $Q u U :\Leftrightarrow U \in N(u)$.

We call $Q$ the neighborhood relation ($Q u U$ reads as “$U$ is a neighborhood of $u$”), and the resulted semantics is called here spatial semantics.
Graphs: Spatial Models

QuU :⇔ u ∈ U for subset models;
QuU :⇔ U ∈ N(u) for neighborhood models.
The classical neighborhood box operator is different from either of $\Box$ and $\mathbf{□}$:

$$
\mathcal{S}, g_0, g_1, u, U \models \Box \varphi \quad \text{iff.} \quad \varphi_{\mathcal{S}, g_0, g_1, U} \in N(u)
$$

$$
\mathcal{S}, g_0, g_1, u, U \models \Diamond \varphi \quad \text{iff.} \quad \mathcal{W} - \varphi_{\mathcal{S}, g_0, g_1, U} \notin N(u),
$$

where $U \in N(u)$ is a neighborhood of the point $u$. 
The Interpretation of $\square$

$\mathcal{S}, g_0, g_1, u, U \models \square \varphi$ iff. $\varphi^{\mathcal{S}, g_0, g_1, U} \in N(u)$

"$u \models \square \varphi$ if and only if the interpretation of $\varphi$ is one of $U$, $V$ or $W$."

The New Language

We enrich our language with neighborhood modalities:

\[ \varphi ::= \top | p | x | X | \neg \varphi | \varphi \land \psi | \Diamond \varphi | \Diamond \Diamond \varphi | \lozenge \varphi | @^X_x \varphi | \downarrow^S_s . \varphi, \]

where \( p \in \text{PROP} \), \( x \in \text{PNT} \), \( X \in \text{SET} \), \( s \in \text{PNTVAR} \), \( S \in \text{SETVAR} \).

We can have an axiomatization based on spatial semantics likewise.
Example

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- The third fact can be expressed in our language:

“The policewoman knows that she would have known whether the car was speeding if her radar gun had the accuracy of ±1 kph.”

- We can talk about belief in two ways:
  - Using the language $\mathcal{H}^2(@, \downarrow)$;
  - Using the enriched language with ♦.
Even More...
Binary Hybrid Operators v.s. Two Unary Ones

@_{x}@_{X}φ is equivalent to @_{x}^X φ if the following hold:

1. @_{x}@_{X}φ ↔ @_{X}@_{x}φ
2. @_{X}@_{Y}φ ↔ @_{Y}@_{X}φ

But if we drop the second condition, which allows a set nominal relying on neighborhood, the neighborhood modality will be easier to be accommodated.
we can add a rule, Name, to cover non-prefixed formulas:

\[
\text{(Name)} \quad \frac{@_x^X \Gamma \rightarrow @_x^X \Delta}{\Gamma \rightarrow \Delta} \quad x, X \text{ new}
\]
A hybrid topological model \((T, \tau, V)\) is a hybrid subset model which satisfies the following conditions:

- \(\emptyset \in \tau, \ T \in \tau;\)
- \(\tau\) is closed under finite intersection and arbitrary union.


Jeremy Seligman.
Internalization: The case of hybrid logics.
Special Issue on Hybrid Logics. Areces, C. and Blackburn, P. (eds.).
感谢各位！

Thanks!