2D Hybrid Logic of Spaces

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A Somewhat Familiar Example

The real speed of a car: 121 kph

Policewoman's radar gun: 121 kph

Accuracy of the radar gun: ± 2 kph

Speed limit of the highway: 120 kph

Fact

- $(Speeding)$
- $\neg K_W(Speeding) \wedge \neg K_W \neg(Speeding)$
- *The policewoman knows that she would have known whether the car was speeding if her radar gun had the accuracy of* ± 1 *kph.*

Logic of Subset Spaces (Moss & Parikh [3])

PROP: propositional variables.

The language:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond\!\!\!\cdot\,\varphi \mid \Diamond\varphi,$$

where $p \in$ PROP.

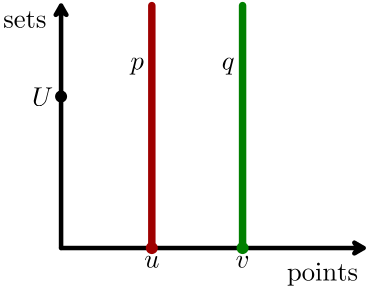
The semantics:

$$\begin{array}{lll} \mathfrak{S}, u, U \models p & \textit{iff.} & u \in V(p) \\ \mathfrak{S}, u, U \models \Diamond\!\!\!\cdot\,\varphi & \textit{iff.} & \exists v \in U.\ \mathfrak{S}, v, U \models \varphi \\ \mathfrak{S}, u, U \models \Diamond\varphi & \textit{iff.} & \exists V.\ (u \in V \subseteq U \ \&\ \mathfrak{S}, u, V \models \varphi), \end{array}$$

where $u \in U$ and $\mathfrak{S} = (S, \Sigma, V)$ is a subset model.

Graphs of the Semantics (I)

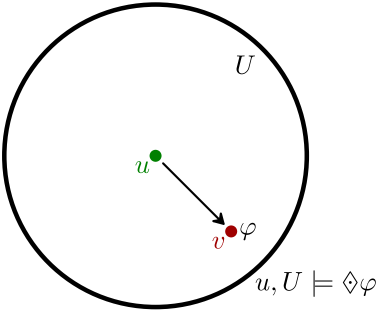
Truth of propositional variables relies only on points.

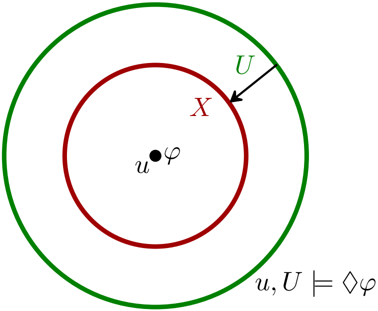


$u, U \models p$
$v, U \models q$

$$\mathfrak{S}, u, U \models p \quad \textit{iff.} \quad u \in V(p)$$

Graphs of the Semantics (II)





$$\mathfrak{S}, u, U \models \dot{\Diamond}\varphi \quad \textit{iff.} \quad \exists v \in U.\ \mathfrak{S}, v, U \models \varphi$$

$$\mathfrak{S}, u, U \models \Diamond\varphi \quad \textit{iff.} \quad \exists X.\ (u \in X \subseteq U \ \&\ \mathfrak{S}, u, X \models \varphi)$$

In the Sense of Epistemic Logic

- $\boxdot$ can be taken as the ordinary K.
- $\Diamond$ operator shrinks the epistemic range, which refines the agent's knowledge. This can be regarded as a sort of *epistemic effort* which is hard to be characterized by the classical epistemic logic.

Issues and Motivations

- Lack of scaling mechanism:
  the third fact in the previous example cannot be expressed;
- Adaptions to talk about belief (reflexivity should not hold):
  - Reinterpreting original modalities (e.g. using the *derived set* operation)
  - Adding new modalities (say, *difference* modality, cf. Kudinov [2])

Solutions or Alternative Ways

- Lack of scaling mechanism:
  We add names for points and sets.
- Adaptions to talk about belief, feelings and so on:
  - *We adopt neighborhood semantics and the neighborhood box operator to cover non-reflexive situations;*
  - *We use $\downarrow$-operator to express the "difference" modality.*

Two-Sorted Hybrid Languages

and

Spatial Semantics

Naming Points and Neighborhoods

New Atoms	NOM	SVAR
PNT	PNTNOM	PNTVAR
SET	SETNOM	SETVAR

$$AT = PROP \cup NOM \cup SVAR$$

Two-Sorted Hybrid Languages

Definition

The language $\mathcal{H}^2(@, \downarrow)$ is given by the following rule:

$$\varphi ::= \top \mid p \mid x \mid X \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond\varphi \mid \Diamond\varphi \mid @_x^X \varphi \mid \downarrow_s^S .\varphi$$

where $p \in \text{PROP},\ x \in \text{PNT},\ X \in \text{SET},\ s \in \text{PNTVAR},\ S \in \text{SETVAR}.$

Spatial Semantics (I): Hybrid Subset Models

Hybrid subset model (S, Σ, V):

- S, a collection of *points*, is the domain;
- $\Sigma \subseteq \wp S$ is a collection of *sets* (or *neighborhoods*);
- $V : \text{PROP} \cup \text{NOM} \to S \cup \wp S$ is an evaluation mapping
  - every propositional variables to a set of points,
  - every point nominal to a point,
  - every set nominal to a set.

Two assignments $g_0 : \text{PNTVAR} \to S$ and $g_1 : \text{SETVAR} \to \Sigma$ are for the two sorts of nominals respectively.

Spatial Semantics (II): Interpreting Hybrid Operators

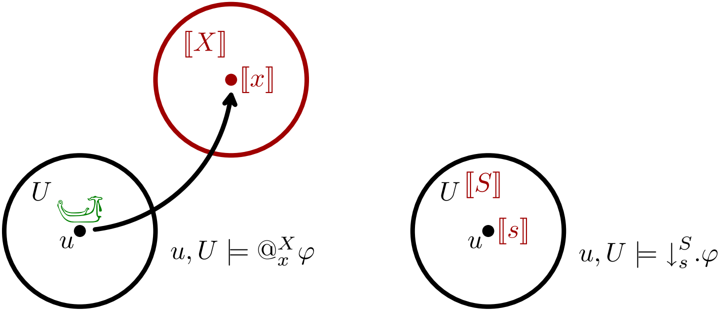
Let $\mathfrak{S} = (S, \Sigma, V)$ be a hybrid subset model, g_0, g_1 two assignments for points and sets. For every point u and a neighborhood U of u,

$$\begin{array}{lll}
\mathfrak{S}, g_0, g_1, u, U \models x & \textit{iff.} & u = x^{\mathfrak{S}, g_0} \\
\mathfrak{S}, g_0, g_1, u, U \models X & \textit{iff.} & U = X^{\mathfrak{S}, g_1} \\
\mathfrak{S}, g_0, g_1, u, U \models @_x^X \varphi & \textit{iff.} & \mathfrak{S}, g_0, g_1, x^{\mathfrak{S}, g_0}, X^{\mathfrak{S}, g_1} \models \varphi \\
\mathfrak{S}, g_0, g_1, u, U \models \downarrow_s^S.\varphi & \textit{iff.} & \mathfrak{S}, g_0[^u_s], g_1[^U_S], u, U \models \varphi
\end{array}$$

where $x \in \text{PNT},\ X \in \text{SET},\ s \in \text{PNTVAR},\ S \in \text{SETVAR}$, and

$$g_0[^u_s](s') = \begin{cases} u, & s = s', \\ g_0(s), & \text{otherwise.} \end{cases}$$ The assignment g_1 is similar.

Graphs of the Semantics



$$\mathfrak{S}, g_0, g_1, u, U \models @^X_x \varphi \quad \textit{iff.} \quad \mathfrak{S}, g_0, g_1, x^{\mathfrak{S}, g_0}, X^{\mathfrak{S}, g_1} \models \varphi$$

$$\mathfrak{S}, g_0, g_1, u, U \models \downarrow^S_s . \varphi \quad \textit{iff.} \quad \mathfrak{S}, g_0[^u_s], g_1[^U_S], u, U \models \varphi$$

Axiomatization

The @-prefixed Gentzen System $G_{\mathcal{H}^2(@,\downarrow)}$

Cf. pg. 201 in the Proceedings for the details.

- Cut is admissible;
- Quasi-subformula property;
- Soundness and completeness.

Internalizing the Semantics (Blackburn[1], Seligman[4])

We express the semantics in a two-sorted first-order language, and then internalize it into a hybrid logic.

$$\mathfrak{S}, g_0, g_1, u, U \models \Diamond\varphi \quad \textit{iff.} \quad \exists V.(u \in V \subseteq U \ \& \ \mathfrak{S}, g_0, g_1, u, V \models \varphi)$$

$$\Downarrow$$

$$R^{\Diamond\varphi} x X \leftrightarrow \exists Y.(x \in Y \subseteq X \ \wedge \ R^{\varphi} x Y)$$

$$\Downarrow$$

$$@_x^X \Diamond\varphi \leftrightarrow \exists Y.(x \in Y \subseteq X \wedge @_x^Y \varphi)$$

Going Further…

Spatial Semantics (III): Hybrid Neighborhood Models

A structure $\mathfrak{M} = (W, N, V)$ is called a *hybrid neighborhood model*, if the following hold:

- $W \neq \emptyset$;
- $N : W \to \wp\wp W$, which is called a *neighborhood function;*
- $V : \text{PROP} \cup \text{NOM} \to W \cup \wp W$, where $V(p) \in \wp W,\ V(a) \in W$.

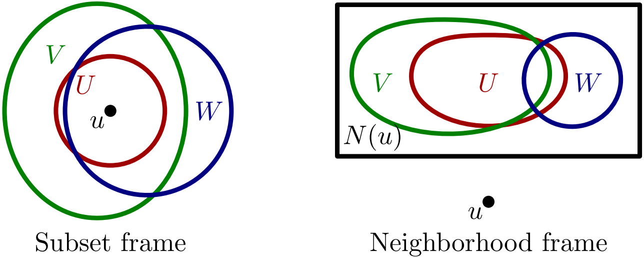
Spatial Semantics (IV): A Unified Semantics

For subset models: define $QuU :\Leftrightarrow u \in U$;

For neighborhood semantics: define $QuU :\Leftrightarrow U \in N(u)$.

We call Q the *neighborhood relation* (QuU reads as "U is a neighborhood of u"), and the resulted semantics is called here *spatial semantics*.

Graphs: Spatial Models



$QuU :\Leftrightarrow u \in U$ for subset models;

$QuU :\Leftrightarrow U \in N(u)$ for neighborhood models.

Neighborhood Modalities

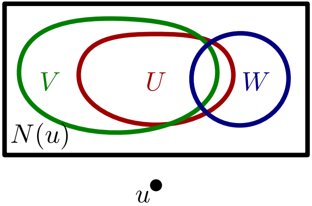
The classical neighborhood box operator is different from either of $\boxdot$ and $\Box$:

$$\begin{array}{lll} \mathfrak{S}, g_0, g_1, u, U \models \blacksquare\varphi & \textit{iff.} & \varphi^{\mathfrak{S}, g_0, g_1, U} \in N(u) \\ \mathfrak{S}, g_0, g_1, u, U \models \blacklozenge\varphi & \textit{iff.} & W - \varphi^{\mathfrak{S}, g_0, g_1, U} \notin N(u), \end{array}$$

where $U \in N(u)$ is a neighborhood of the point u.

The Interpretation of ■

$$\mathfrak{S}, g_0, g_1, u, U \models \blacksquare\varphi \quad \textit{iff.} \quad \varphi^{\mathfrak{S}, g_0, g_1, U} \in N(u)$$



“$u \models \blacksquare\varphi$ if and only if the interpretation of φ is one of U, V or W.”

The New Language

We enrich our language with neighborhood modalities:

$$\varphi ::= \top \mid p \mid x \mid X \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamonddot\varphi \mid \Diamond\varphi \mid \blacklozenge\varphi \mid @_x^X\varphi \mid \downarrow_s^S.\varphi,$$

where $p \in \textsc{prop},\ x \in \textsc{pnt},\ X \in \textsc{set},\ s \in \textsc{pntvar},\ S \in \textsc{setvar}.$

We can have an axiomatization based on spatial semantics likewise.

Back to the Beginning

Example

The real speed of a car: 121 kph	Policewoman's radar gun: 121 kph
Accuracy of the radar gun: ± 2 kph	Speed limit of the highway: 120 kph

- The third fact can be expressed in our language:

"The policewoman knows that she would have known whether the car was speeding if her radar gun had the accuracy of ± 1 kph." $@_{121}^{(120,122)} \boxdot_W (\mathit{speeding})$

- We can talk about belief in two ways:
  - Using the language $\mathcal{H}^2(@, \downarrow)$;
  - Using the enriched language with $\blacklozenge$.

EVEN MORE...

Binary Hybrid Operators v.s. Two Unary Ones

$@_x @_X \varphi$ is equivalent to $@_x^X \varphi$ if the following hold:

1. $@_x @_X \varphi \leftrightarrow @_X @_x \varphi$
2. $@_X @_Y \varphi \not\leftrightarrow @_Y @_X \varphi$

But if we drop the second condition, which allows a set nominal relying on neighborhood, the neighborhood modality will be easier to be accommodated.

Shifting between Non-@-Prefixed Rules

we can add a rule, Name, to cover non-prefixed formulas:

$$\text{(Name)} \quad \frac{@_x^X \Gamma \longrightarrow @_x^X \Delta}{\Gamma \longrightarrow \Delta} \quad x, X \text{ new}$$

Spatial Semantics (V): Hybrid Topological Models

A *hybrid topological model* (T, τ, V) is a hybrid subset model which satisfies the following conditions:

- $\emptyset \in \tau, \; T \in \tau$;
- τ is closed under finite intersection and arbitrary union.

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感谢各位！

Thanks!