A Study of a Logic for Multiple-source Approximation Systems

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1. Basic Concepts Related to Rough Set Theory

2. Multiple-source Approximation Systems
   - Different Notions of Lower/Upper Approximations
   - Different Notions of Definability

3. Logic for MSAS(LMSAS)
   - Axiomatization
   - Some Decidable Problems
   - Bisimulation
   - Relationship with Other Logics

4. Conclusion
Out break of some disease.

- **Totally unaffected** class → classes 1 to 8.
- **Totally affected** class → class 9.
- **Partially affected** class → class 10.

**Q.** Is ‘A’, a class 8 student, affected?
**A.** No.

**Q.** Is ‘B’, a class 9 student, affected?
**A.** Yes.

**Q.** Is ‘C’, a class 10 student, affected?
**A.** Possibly, but not certainly.
Pawlak Approximation space [Pawlak’82]

\[(U, R)\], where \(R\) is an equivalence relation on \(U\).
Pawlak Approximation space [Pawlak’82]

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Pawlak Approximation space [Pawlak’82]

$(U, R)$, where $R$ is an equivalence relation on $U$. 
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\begin{itemize}
\item $U := \text{Set of students}$;
\item $aRb$ iff $a$ and $b$ are in the same class;
\item $X := \text{Set of affected students}$.
\end{itemize}

\begin{table}
\begin{tabular}{|c|c|c|c|c|}
\hline
$U$ & 1 & 2 & 3 & 4 & 5 \\
\hline
6 & 7 & 8 & 9 & 10 \\
\hline
\end{tabular}
\end{table}
- $U := \text{Set of students}$;
- $aRb$ iff $a$ and $b$ are in the same class;
- $X := \text{Set of affected students}$.
- $U := \text{Set of students};$
- $aRb \iff a \text{ and } b \text{ are in the same class};$
- $X := \text{Set of affected students}.$
Multiple-source Approximation Systems (MSAS)

\[ \mathcal{F} := (U, \{R_i\}_{i \in \mathbb{N}}), \] where

- \( U \) is a non-empty set,
- \( \mathbb{N} \) an initial segment of the set of positive integers, and
- each \( R_i, i \in \mathbb{N} \), is an equivalence relation on the domain \( U \).

\[ |\mathbb{N}| \text{ is referred to as the cardinality of } \mathcal{F} \text{ and is denoted by } |\mathcal{F}|. \]
\( \mathfrak{F} := (U, \{ R_i \}_{i \in \mathbb{N}}), \ X \subseteq U \)

**Definition**

- **Strong lower approximation**
  \[ X_s := \bigcap_i X_{R_i}; \]

- **Weak lower approximation**
  \[ X_w := \bigcup_i X_{R_i}; \]

- **Strong upper approximation**
  \[ \overline{X_s} := \bigcap_i \overline{X}_{R_i}; \]

- **Weak upper approximation**
  \[ \overline{X_w} := \bigcup_i \overline{X}_{R_i}; \]

For MSAS \( \mathfrak{F} := (U, \{ R \}) \)
\[ X_s = X_w = X_R \text{ and } \overline{X_s} = \overline{X_w} = \overline{X_R} \]
\[ X_s \subseteq X_w \subseteq X \subseteq X_s \subseteq X_w \]
$X_s \subseteq X_w \subseteq X \subseteq \overline{X_s} \subseteq \overline{X_w}$
Proposition

1. \( X \cap Y_s = X_s \cap Y_s; \quad X \cup Y_w = X_w \cup Y_w; \)
2. \( X \cap Y_s \subseteq X_s \cap Y_s; \quad X \cup Y_w \supseteq X_w \cup Y_w; \)
3. \( X^c_s = (X_w)^c; \quad X^c_w = (X_s)^c; \)
4. \( X^c_s = (X_w)^c; \quad X^c_w = (X_s)^c; \)
5. \( X_w = \overline{(X_w)_w}; \quad \overline{X_s} = (\overline{X_s})_s; \)
6. \( \overline{X_w} = (\overline{X_w})_w = (\overline{X_s})_w; \)
7. \( (\overline{X_s})_w \subseteq X_w; \)
\( X \subseteq U \) is **lower definable** if \( X_s = X_w \).

\[ U \]

\[ (X_w)^c \]

\[ X_w \setminus X_s \]

\[ X_s \setminus X_w \]

\[ X_w \setminus X_s \]

\[ X_s \]

- **certain +ve**
- **possible +ve**
- **certain boundary**
- **possible -ve**
- **certain -ve**

\( X \) is lower definable iff the sets of +ve elements in all approximations spaces are identical.
\( X \subseteq U \) is upper definable if \( \overline{X}_s = \overline{X}_w \).

\[ U \]

\[ (\overline{X}_w)^c \]

\[ \overline{X}_w \setminus \overline{X}_s \]

\[ \overline{X}_s \setminus \overline{X}_w \]

\[ \overline{X}_w \setminus \overline{X}_s \]

\[ X_s \]

- certain +ve
- possible +ve
- certain boundary
- possible -ve
- certain -ve

\( X \) is upper definable iff the sets of -ve elements in all approximations spaces are identical.
\[ X \subseteq U \text{ is weak definable if } \overline{X}_s = \underline{X}_w. \]

\[ X \text{ is weak definable iff } X \text{ does not have certain boundary element.} \]
**X ⊆ U** is strong definable if \( X_s = X_w \).

\[ X_w \setminus X_s \]

\[ X_s \setminus X_w \]

\[ X_w \setminus X_s \]

\[ X_s \]

- **certain +ve**
- **possible +ve**
- **certain boundary**
- **possible -ve**
- **certain -ve**

\( X \) is strong definable iff every element of \( U \) is either certain +ve or certain -ve.
Proposition

- $X$ is upper definable iff $X^c$ is lower definable.

- Arbitrary union (intersection) of upper (lower) definable sets is also upper (lower) definable.
  (Collection of upper (lower) definable sets is not closed under intersection (union)).

- Collection of all strong definable sets forms a complete field of sets.
Proposition

The following are equivalent:

- $X$ is strong definable.
- $X$ is both lower and upper definable and $X$ is definable in some approximation space.
- $X$ is definable in each approximation space.
- $X_s = X_w = X = X_s = X_w$. 
Language \( \mathcal{L} \)

- a non-empty countable set \( \text{Var} \) of variables,
- a (possibly empty) countable set \( \text{Con} \) of constants,
- a non-empty countable set \( \text{PV} \) of propositional variables and
- the propositional constants \( \top, \bot \).

Terms \( T \) := \( \text{Var} \cup \text{Con} \).

Wffs:= \( \top | \bot | p | \neg \alpha | \alpha \land \beta | \langle t \rangle \alpha | \forall x \alpha \)

\( p \in \text{PV}, \ t \in T, \ x \in \text{Var}, \) and \( \alpha, \beta \) are wffs.
### Notations

- $\mathcal{F} \longrightarrow$ Set of all wffs;
- $\overline{\mathcal{F}} \longrightarrow$ Set of all closed wffs;
- $\text{Con}(\alpha) \longrightarrow$ Set of constants used in $\alpha$;
- $\text{Var}(\alpha) \longrightarrow$ Set of variables used in $\alpha$;
- $\text{FV}(\alpha) \longrightarrow$ Set of free variables used in $\alpha$. 
Model

\( \mathcal{M} := (\bar{\mathcal{F}}, V, \nu) \), where

- \( \bar{\mathcal{F}} := (U, \{R_i\}_{i \in \mathbb{N}}) \) is a MSAS;
- \( V : P V \rightarrow 2^U \) is a valuation function and
- \( \nu : Var \rightarrow \mathbb{N} \) is an assignment.

Let \( \alpha \in \mathcal{F} \) and \( \Gamma \subseteq \mathcal{F} \).

\( \alpha \)-Model

The model \( \mathcal{M} := (\bar{\mathcal{F}}, V, \nu) \) is said to be an \( \alpha \)-model if \( |\bar{\mathcal{F}}| \geq k \), where \( k \) is the largest integer such that \( c_k \in Con(\alpha) \).

\( \Gamma \)-Model

\( \mathcal{M} \) is a \( \Gamma \)-model, if it is an \( \alpha \)-model for each \( \alpha \in \Gamma \).
satisfiability

- $\mathcal{M}, w \models p$, if and only if $w \in V(p)$.
- $\mathcal{M}, w \models \langle c_i \rangle \alpha$, if and only if there exists $w'$ in $U$ such that $wR_i w'$ and $\mathcal{M}, w' \models \alpha$.
- $\mathcal{M}, w \models \langle x \rangle \alpha$, if and only if there exists $w'$ in $U$ such that $wR_{v(x)} w'$ and $\mathcal{M}, w' \models \alpha$.
- $\mathcal{M}, w \models \forall x \alpha$, if and only if $\mathcal{M}', w \models \alpha$, for every model $\mathcal{M}' := (\mathcal{F}, V, v')$ where the assignment $v'$ is $x$-equivalent to $v$. 

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Validity

\[ \models \alpha, \text{ if and only if } M, w \models \alpha, \text{ for every } \alpha\text{-model } M \text{ and object } w \text{ of } U. \]
\[ \mathcal{M} := (\mathcal{F}, V, \nu) \rightarrow \text{a model.} \]

\[ V(\alpha) := \{ w \in U : \mathcal{M}, w \models \alpha \}, \text{ where } \alpha \text{ is a wff involving only those } c_k \text{ which satisfy } |\mathcal{F}| \geq k \]

---

**Rough Set Interpretation**

- \( V(\langle c_i \rangle \alpha) = \overline{V(\alpha)}_{R_i} ; \)
- \( V([c_i] \alpha) = \overline{V(\alpha)}_{R_i} ; \)
- Fore \( \alpha \) which does not have free occurrence of \( x \),
  - \( V(\forall x[x] \alpha) = \overline{V(\alpha)}_{s} ; \)
  - \( V(\exists x[x] \alpha) = \overline{V(\alpha)}_{w} ; \)
  - \( V(\forall x\langle x \rangle \alpha) = \overline{V(\alpha)}_{s} ; \)
  - \( V(\exists x\langle x \rangle \alpha) = \overline{V(\alpha)}_{w} . \)
Proposition

The following are valid.

1. (a) $\exists x[x] \alpha \rightarrow \alpha$.
   (b) $\alpha \rightarrow \forall x \langle x \rangle \alpha$.

2. (a) $\forall x[x](\alpha \land \beta) \leftrightarrow \forall x[x] \alpha \land \forall x[x] \beta$.
   (b) $\exists x \langle x \rangle (\alpha \lor \beta) \leftrightarrow \exists x \langle x \rangle \alpha \lor \exists x \langle x \rangle \beta$.

3. (a) $\forall x \langle x \rangle (\alpha \land \beta) \rightarrow \forall x \langle x \rangle \alpha \land \forall x \langle x \rangle \beta$.
   (b) $\exists x[x](\alpha \lor \beta) \rightarrow \exists x[x] \alpha \lor \exists x[x] \beta$.

4. (a) $\exists x[x] \alpha \leftrightarrow \exists x[x] \exists y[y] \alpha$.
   (b) $\exists x \langle x \rangle \forall y[y] \alpha \rightarrow \exists x[x] \alpha$. 

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Axioms

1. All axioms of classical propositional logic.
2. $\forall x \alpha \rightarrow \alpha[t/x]$, where $\alpha$ admits the term $t$ for the variable $x$.
3. $\forall x (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \forall x \beta)$, where the variable $x$ is not free in $\alpha$.
4. $\forall x[t] \alpha \rightarrow [t] \forall x \alpha$, where the term $t$ and variable $x$ are different.
5. $[t](\alpha \rightarrow \beta) \rightarrow ([t] \alpha \rightarrow [t] \beta)$.
6. $\alpha \rightarrow \langle t \rangle \alpha$.
7. $\alpha \rightarrow [t] \langle t \rangle \alpha$.
8. $\langle t \rangle \langle t \rangle \alpha \rightarrow \langle t \rangle \alpha$. 
Rules of inference

\[
\forall. \quad \frac{\alpha}{\forall x \alpha} \\
\text{MP.} \quad \frac{\alpha}{\alpha \rightarrow \beta} \\
\text{N.} \quad \frac{\alpha}{[t] \alpha}
\]
Soundness

If $\not\vdash \alpha$ then $\models \alpha$

Completeness Theorem

If $\models \alpha$ then $\vdash \alpha$

Proof

- Let $n$ is the largest integer such that $c_n$ occurs in $\alpha$.
- $\mathcal{L}^n$ obtained from the language $\mathcal{L}$ by restricting Con to the set $\{c_1, c_2, \ldots, c_n\}$.
- $\mathcal{L}^{n+}$ obtained from the language $\mathcal{L}^n$ by adding infinitely many new variables.
- $\text{Var}^+ := \{x_1, x_2, \ldots\}$ be an enumeration of the variables in $\mathcal{L}^{n+}$. 
Proof (Contd.)

Canonical model:

- $U^C := \{ w : w$ is maximally consistent and has the $\forall$ property in $L^{n+}\}$.
- $V^C : PV \rightarrow \mathcal{P}(U^C)$ is such that $V^C(p) := \{ w \in U^C : p \in w \}$, for $p \in PV$.
- $\nu^C : Var^+ \rightarrow \mathbb{N}$ is such that $\nu^C(x_i) := n + i$.
- $wR^C_i w'$, if and only if for every wff $[t] \alpha$ of $L^{n+}$ with $\nu^C(t) = i$, $[t] \alpha \in w$ implies $\alpha \in w'$, where $w, w' \in U^C$.
- $\mathcal{F}^C := (U^C, \{ R^C_i \}_{i \in \mathbb{N}})$.
- $\mathcal{M}^C := (\mathcal{F}^C, V^C, \nu^C)$.
Proof (Contd.)

Proposition

Every consistent set of wffs in $\mathcal{L}^n$ has a maximally consistent extension in $\mathcal{L}^{n+}$, having the $\forall$-property.

Truth Lemma

For any wff $\beta$ in $\mathcal{L}^{n+}$ and $w \in U^C$, $\beta \in w$ if and only if $M^C, w \models \beta$.

- $\nu^c(x) \neq \nu^c(y)$ for $x \neq y$;
- $\nu^c(x) \not\in \{1, 2, \ldots, n\}$. 
Some Decidable problems

Proposition

Given a wff \( \alpha \) and an integer \( m \geq k \), where \( k \) is the largest integer such that \( c_k \in Con(\alpha) \), it is decidable if there exists an \( \alpha \)-model \( M := (\mathcal{F}, V, v) \) with \( |\mathcal{F}| = m \) such that \( \alpha \) is satisfiable in \( M \).

Proof

- \( \Sigma \xrightarrow{\text{Sub-formula closed set}} \) Sub-formula closed set such that \( r \) is the largest integer for which \( c_r \in Con(\Sigma) \).
- \( \mathcal{F} := (U, \{R_i\}_{1 \leq i \leq m}), \ m \geq r. \)
- \( V : PV \rightarrow 2^U. \)
Proof(Contd.)

- For $w, w' \in U$, $w \equiv_\Sigma w'$, if and only if for all $\beta \in \Sigma$ and all $\Sigma$-models $M := (\mathcal{F}, V, \nu)$,

$$M, w \models \beta \iff M, w' \models \beta$$

Filtration Model

- $U^f := \{[w] : w \in U\}$.
- $[w] R_i^f [w']$ if and only if there exist $w_1 \in [w]$ and $w_2 \in [w']$ such that $w_1 R_i w_2$;
- $R_i^{f*}$ is the transitive closure of $R_i^f$.
- $\mathcal{F}^f := (U^f, \{R_i^{f*}\}_{1 \leq i \leq m})$.
- $V^f(p) := \{[w] \in U^f : w \in V(p)\}$. 
Proof (Contd.)

Filtration Theorem

For all wffs $\beta \in \Sigma$, all assignment $\nu : \text{Var} \rightarrow \{1, 2, \ldots, m\}$ and all objects $w \in U$,

$$(\mathcal{F}, \mathcal{V}, \nu), w \models \beta \text{ iff } (\mathcal{F}^f, \mathcal{V}^f, \nu), [w] \models \alpha$$
Proof (Contd.)

Proposition

Let $\alpha$ be a wff such that

- $\text{Var}(\alpha) := \{x_1, x_2, \ldots, x_n\}$.
- $\alpha$ is satisfiable in a model $\mathcal{M} := (\mathcal{F}, V, \nu)$, where $|\mathcal{F}| = m$.
- $\Sigma$ is the set of all sub-formulae of $\alpha$.

Then $\alpha$ is satisfiable in a model with domain of cardinality $\leq 2|\Sigma| \times m^n$.

- $\text{Asg} := \{\nu \mid \nu : \text{var} \to \{1, 2, \ldots, m\}\}$.
- For $\nu_1, \nu_2 \in \text{Asg}$, $\nu_1 \approx \nu_2$ iff $\nu_1(x) = \nu_2(x)$, $x \in \text{Var}(\alpha)$.
- $|\text{Asg}/\approx| \leq m^n$. 
Proof(Contd.)

- The mapping \( g : U^f \rightarrow 2^{\Sigma \times \text{Asg}/\approx} \), defined as 
  \[
g([w]) := \{(\beta, [v_1]) \in \Sigma \times \text{Asg}/\approx : ((\mathcal{F}, V, v_1), w \models \beta)\}
\]
is injective.

- \( |U^f| \leq 2|\Sigma| \times m^n \)
Proposition

Given an integer $t$ and a wff $\alpha$, it is decidable if there exists an $\alpha$-model with a domain of cardinality $t$, in which $\alpha$ is satisfiable.
\[ R \subseteq W \times W \quad \text{and} \quad R' \subseteq W' \times W'. \quad Z \subseteq W \times W' \]

**Bisimulation**

\[ Z : R \leftrightarrow R' \], if the following conditions are satisfied:
$R \subseteq W \times W$ and $R' \subseteq W' \times W'$. $Z \subseteq W \times W'$

**Bisimulation**

$Z : R \leftrightarrow R'$, if the following conditions are satisfied:
\( R \subseteq W \times W \) and \( R' \subseteq W' \times W' \). \( Z \subseteq W \times W' \)

**Bisimulation**

\( Z : R \leftrightarrow R' \), if the following conditions are satisfied:

\[
\begin{align*}
A & \quad w & \quad Z & \quad w' \\
R & \quad u & & \quad u'
\end{align*}
\]
$R \subseteq W \times W$ and $R' \subseteq W' \times W'$. $Z \subseteq W \times W'$

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**Bisimulation**

\[ Z : R \leftrightarrow R', \text{ if the following conditions are satisfied:} \]

- \( R \subseteq W \times W \) and \( R' \subseteq W' \times W' \).
- \( Z \subseteq W \times W' \).
- For all \( w \in W \), if \( (w, u) \in R \), then there exists \( w' \) in \( W' \) such that \( (w, w') \in Z \) and \( (w', u') \in R' \).
- For all \( u' \in W' \), if \( (u, u') \in R' \), then there exists \( w \) in \( W \) such that \( (w', w) \in Z \) and \( (u', u') \in R' \).
\[ C \subseteq \text{Con}, \ V^1, V^2 \subseteq \text{Var}. \]
\[ \Gamma := \{ \alpha : \text{Con}(\alpha) \subseteq C, \ FV(\alpha) \subseteq V^1 \text{ and } \text{Var}(\alpha) \subseteq V^2 \}. \]

**Theorem**

\[ \mathcal{M} := (\mathcal{F}, V, \nu) \quad \mathcal{M}' := (\mathcal{F}', V', \nu') \]
\[ Z \subseteq W \times W' \]
\[ C \subseteq \text{Con}, \ V^1, V^2 \subseteq \text{Var}. \]
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$M := (\mathcal{F}, V, v)$ \quad $M' := (\mathcal{F}', V', v')$

$Z \subseteq W \times W'$

$C \subseteq \text{Con}$, $V^1$, $V^2 \subseteq \text{Var}$.
\[ C \subseteq Con, \ V^1, V^2 \subseteq Var. \]
\[ \Gamma := \{ \alpha : Con(\alpha) \subseteq C, \ FV(\alpha) \subseteq V^1 \text{ and } Var(\alpha) \subseteq V^2 \} \].

**Theorem**

\[ M := (\mathcal{F}, V, v) \quad M' := (\mathcal{F}', V', v') \]
\[ Z \subseteq W \times W' \]

\[ Z : R_i \leftrightarrow R'_i \]
\( C \subseteq \text{Con}, \ V^1, V^2 \subseteq \text{Var}. \)
\[ \Gamma := \{ \alpha : \text{Con}(\alpha) \subseteq C, \ \text{FV}(\alpha) \subseteq V^1 \ \text{and} \ \text{Var}(\alpha) \subseteq V^2 \}. \]

**Theorem**

\[
\mathcal{M} := (\mathcal{F}, V, \nu) \quad \text{and} \quad \mathcal{M}' := (\mathcal{F}', V', \nu')
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\[ Z \subseteq W \times W' \]

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**Theorem**

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\[ Z \subseteq W \times W' \]

\[ Z : R_i \leftrightarrow R'_i \]

\[ R_{\nu(x)} \quad R'_{\nu'(x)} \]
$C \subseteq \text{Con}, \ V^1, V^2 \subseteq \text{Var}.$
$\Gamma := \{ \alpha : \text{Con}(\alpha) \subseteq C, \ FV(\alpha) \subseteq V^1 \text{ and } \text{Var}(\alpha) \subseteq V^2 \}. $

**Theorem**

$M := (\mathcal{F}, V, v) \quad M' := (\mathcal{F}', V', v')$

$Z \subseteq W \times W'$

$Z : R_i \leftrightarrow R'_i$

$Z : R_{v(x)} \leftrightarrow R'_{v'(x)}$
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Proof:

\[ V^2 \]

\[ C \]

\[ x \]

\[ R_v(x) \]
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Outline
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Conclusion

Axiomatization
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Relationship with Other Logics

\[ Z : R_v(x) \iff R'_j \]

\[ R_j \iff R'_{v'}(x) \]
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\[ V^2 \]

\[ C \]
\[ x \]

\[ D \]

\[ Z : R_v(x) \overset{\leftrightarrow}{\rightarrow} R'_j \]

\[ Z : R_j \overset{\leftrightarrow}{\rightarrow} R'_{v'(x)} \]

\[ u Z u' \]
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\[ Z : R_v(x) \iff R'_j \]

\[ Z : R_j \iff R'_{v'(x)} \]

\[ u \in V(p) \iff u' \in V'(p) \text{ for all } p \in V(p) \]
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Then for all \( \alpha \in \Gamma \),

\[
M, w \models \alpha \iff M', w' \models \alpha.
\]
\( C \subseteq \text{Con}, \ V^2 \subseteq \text{Var}. \)

\[ \Gamma := \{ \alpha : \text{Con}(\alpha) \subseteq C, \ \text{Var}(\alpha) \subseteq V^2 \}. \]

**Corollary(\(*\))**

1. \( \mathcal{M} := (\mathcal{F}, V, \nu) \) and \( \mathcal{M}^\prime := (\mathcal{F}^\prime, V^\prime, \nu^\prime) \), where
   \[ \mathcal{F} := (W, \{ R_i \}_{i \in N}), \ \mathcal{F}^\prime := (W^\prime, \{ R_i^\prime \}_{i \in N^\prime}) \] be two \( \Gamma \)-models.
2. \( Z \subseteq W \times W^\prime \) satisfying the following:
   a. \( Z : R_i \leftrightarrow R_i^\prime \) for all \( i \) such that \( c_i \in C \);
   b. \( Z : R_{\nu(x)} \leftrightarrow R_{\nu'(x)}^\prime \) for all \( x \in V^2 \);
   c. If \( uZu^\prime \), then \( u \in V(p) \) if and only if \( u^\prime \in V'(p) \) for all \( p \in PV \);
   d. \( wZw^\prime \).

Then for all \( \alpha \in \Gamma \),

\[ \mathcal{M}, w \models \alpha \iff \mathcal{M}^\prime, w^\prime \models \alpha. \]
Converse of Corollary (*)?

Let $C \subseteq \text{Con}$, $V^1 \subseteq \text{Var}$. Let $\Gamma := \{ \alpha : \text{Con}(\alpha) \subseteq C, \text{Var}(\alpha) \subseteq V^1 \}$

**Theorem**

1. $\mathcal{M} := (\mathcal{F}, V, v)$ and $\mathcal{M}' := (\mathcal{F}', V', v')$, where $\mathcal{F} := (W, \{ R_i \}_{i \in N})$, $\mathcal{F}' := (W', \{ R'_i \}_{i \in N'})$, be two $\Gamma$-models.

2. Equivalence classes of the relations $R_i, R'_i$ for $c_i \in C$ and $R_v(x), R'_v(x)$, $x \in V^1$ are all finite.

3. $w \in W$ and $w' \in W'$ such that $\mathcal{M}, w \models \alpha \iff \mathcal{M}', w' \models \alpha$ for all $\alpha \in \Gamma$. 
Then there exists a relation $Z \subseteq W \times W'$ satisfying the following:

a. $Z : R_i \leftrightarrow R'_i$ for all $i$ such that $c_i \in C$;

b. $Z : R_{v(x)} \leftrightarrow R'_{v'(x)}$ for all $x \in V^1$;

c. If $uZu'$, then $u \in V(p)$ if and only if $u' \in V'(p)$ for all $p \in PV$;

d. $wZw'$.

- Result does not hold if condition (2) is removed.
$P \subseteq_{f} PV$ and $\alpha$ be a wff which involves only the propositional variables from the set $P$.

**Proposition**

\[ M := (\mathcal{F}, V, v) \quad M' := (\mathcal{F}', V', v') \]

\[ f : W \rightarrow W' \text{ (Surjective)} \]
$P \subseteq_f PV$ and $\alpha$ be a wff which involves only the propositional variables from the set $P$.

**Proposition**

$$M := (\mathcal{F}, V, \nu) \quad M' := (\mathcal{F}', V', \nu')$$

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\]

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\[
Con(\alpha) \\
A \quad c_i \\
\quad \quad s \: R_i \: r \Leftrightarrow f(s) \: R' \: f(r)
\]
$P \subseteq_{f} PV$ and $\alpha$ be a wff which involves only the propositional variables from the set $P$.

<table>
<thead>
<tr>
<th>Proposition</th>
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<tbody>
<tr>
<td>$\mathcal{M} := (\mathcal{F}, V, v)$</td>
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<tr>
<td>$f : W \rightarrow W'$ (Surjective)</td>
</tr>
</tbody>
</table>

### Con($\alpha$)

- $s R_i r \iff f(s) R' f(r)$

### FV($\alpha$)

- $A$
  - $c_i$
- $B$
  - $x$
$P \subseteq_f PV$ and $\alpha$ be a wff which involves only the propositional variables from the set $P$.

\[ M := (\mathcal{F}, V, \nu) \quad M' := (\mathcal{F}', V', \nu') \]

\[ f : W \rightarrow W' \text{ (Surjective)} \]

\[ \text{Con}(\alpha) \]

\[ s \ R_i \ r \iff f(s) \ R' \ f(r) \]

\[ FV(\alpha) \]

\[ s \ R_{v(x)} \ r \iff f(s) \ R'_{v'(x)} \ f(r) \]
Then we have,

\[ M, w \models \alpha \iff M', w' \models \alpha. \]
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$$\mathcal{M}, w \models \alpha \Leftrightarrow \mathcal{M}', w' \models \alpha.$$
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Then we have,

\[ M, w \models \alpha \iff M', w' \models \alpha. \]
Mohua Banerjee, Md. Aquil Khan

A Study of a Logic for Multiple-source Approximation Systems

Then we have,

\( M, w \models \alpha \iff M', w' \models \alpha. \)
Example

\[ W := \{w_i : i \in \mathbb{N}\} \cup \{w, d\} \]
\[ U := \{u_i : i \in \mathbb{N}\} \cup \{u\} \]
\[ R_i = W \times W \]
\[ R'_i = U \times U \]
\[ V(p_j) := W \setminus \{w, w_j\} \]
\[ V(p_j) := U \setminus \{u, u_j\} \]

**Obs 1:** \( d \in V(p) \) for all \( p \in PV \).

**Obs 2:** For each \( P \subseteq_f PV \), there exists a \( t \in U \) such that \( t \in V'(p) \) for all \( p \in P \).

**Claim 1:** \( M, w \models \alpha \iff M', u \models \alpha \) for all \( \alpha \).

- \( P = \{p_{j_1}, p_{j_1}, \ldots, p_{j_n}\} \) consists of propositional variables which occurs in \( \alpha \).
- \( f(d) = u_{j_n+1}, f(w) = u, f(w_i) = u_i \).
- Use previous result.
Obs3: There is no $t \in U$ such that $t \in V'(p)$ for all $p \in PV$.

Claim 2: There is $Z$ satisfying (a)-(d).

- If possible, let there exists such a $Z$
- $wRu$ and $wZu$.
- There exists $t \in U$ such that $dZt$.
- $t \in V'(p)$ for all $p \in PV$. 
Bisimulation Invariance Result and the Hennessy-Milner Theorem for $S_5$

- $\Gamma := \{ \alpha : \text{Con}(\alpha) \subseteq \{c_1\}, \text{Var}(\alpha) = \emptyset \}$ corresponds to the set of all wffs of a normal modal logic, where $[c_1]$ and $\langle c_1 \rangle$ are considered as $\Box$ and $\Diamond$ respectively.

- $\Phi : (W, \{R_i\}_{i \in \mathbb{N}}) \mapsto (W, R_1)$ (**Surjective**).

- $(\mathcal{F}, V, v), w \vDash \alpha$ if and only if $(\Phi(\mathcal{F}), V), w \vDash_{S_5} \alpha$, for all $\alpha \in \Gamma$.

- From this observation and choice of $\Gamma$, we obtain the bisimulation invariance result and the Hennessy-Milner theorem for $S_5$. 
Proposition

- $S5 \rightarrow LMSAS$.
- $K_n \rightarrow LMSAS$.
- $LMSAS \rightarrow SOL$.
- $B \rightarrow LMSAS$.

Proof

- Let $L$ be the set of basic modal logic wffs with modal operator $L$.
- Choose a variable $x$ and fix it.
- $T_B : L \rightarrow \mathcal{L}$ be the translation such that:
  \[ T_B(L\alpha) = \forall x[x] T_B(\alpha). \]
Proof (Contd.)

Proposition

$\alpha \in L$ is satisfiable in a reflexive, symmetric model if and only if $T_B(\alpha)$ is a satisfiable LMSAS wff.

- $\mathfrak{F} := (W, \{R_i\}_{i \in N})$, $\mathfrak{F}^* := (W, R := \bigcup_i R_i)$, where $R$ is reflexive and symmetric.

- $(\mathfrak{F}, V, v), w \models T_B(\alpha)$ iff $(\mathfrak{F}^*, V), w \models \alpha$

- Given reflexive and symmetric frame $(W, R)$ with finite domain, there exists MSAS $\mathfrak{F}$ such that $\mathfrak{F}^* = (W, R)$. 

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Decidability?

1. There exists a function $f$ such that if $\alpha$ is satisfiable, then it is satisfiable in a model based on a MSAS $\mathcal{F}$ with $|\mathcal{F}| \leq f(|\alpha|)$.

2. There exists a function $f$ such that if $\alpha$ is satisfiable, then it is satisfiable in a model with finite domain $W$ such that $|W| \leq f(|\alpha|)$.

LMSAS corresponds to which fragment of SOL?

Extension of LMSAS to capture the group knowledge of the sources.
Thank you