Intriguing Graph Polynomials

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Reporting also recent work by M. Freedman, L. Lovász, A. Schrijver and B. Szegedy

Graph polynomial project:
http://www.cs.technion.ac.il/~janos/RESEARCH/gp-homepage.html

Overview

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- Evaluations of graph polynomials
- What we find intriguing
- Numeric graph invariants: Properties and guiding examples
- Connection matrices
- **MSOL**-definable graph polynomials
- Finite rank of connection matrices
- Applications of the Finite Rank Theorem
- Complexity of evaluations of graph polynomials
- Towards a dichotomy theorem

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Parametrized numeric graph invariants

The (vertex) chromatic polynomial

Let $G = (V(G), E(G))$ be a graph, and $\lambda \in \mathbb{N}$.

A λ-**vertex-coloring** is a map

$$c : V(G) \to [\lambda]$$

such that $(u, v) \in E(G)$ implies that $c(u) \neq c(v)$.

We define $\chi(G, \lambda)$ to be the number of λ-vertex-colorings

Theorem: (G. Birkhoff, 1912)

$\chi(G, \lambda)$ is a polynomial in $\mathbb{Z}[\lambda]$.

Proof:

(i) $\chi(E_n) = \lambda^n$ where E_n consists of n isolated vertices.

(ii) For any edge $e = E(G)$ we have $\chi(G - e, \lambda) = \chi(G, \lambda) + \chi(G/e, \lambda)$.

Parametrized numeric graph invariants

A **parametrized numeric graph invariant** is a function

$$f : \mathcal{G} \times R \rightarrow \mathbb{R}$$

which is invariant under graph isomorphism.
Here R can be $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ or any ring.

Examples:

(i) $ind_k(G)$ the number of independent sets of size k.

(ii) $ind(G, X) = \sum_i ind(G, i) \cdot X^i$, the independent set polynomial.

(iii) The chromatic polynomial $\chi(G, \lambda)$.

(iv) Any graph polynomial from the literature, like matching polynomials, Tutte polynomial, interlace polynomial, cover polynomial of directed graphs, etc.

Coding many graph parameters into a graph polynomial

A particular graph polynomial is considered

interesting

if it encodes many useful graph parameters.

The characteristic polynomial

Let $G = (V(G), E(G))$ be a graph.

The characteristic polynomial $P(G, X)$ is defined as the characteristic polynomial (in the sense of linear algebra) of the adjacency matrix A_G of G defined as

$$det(X \cdot \mathbf{1} - A_G) = \sum_{i=0}^{n} c_i(G) \cdot X^{n-i}.$$

It is well known that

(i) $n = |V(G)|$

(ii) $-c_2(G) = |E(G)|$

(iii) $-c_3(G)$ equals twice the number of triangles of G.

(iv) The second largest zero $\lambda_2(G)$ of $P(G; X)$ gives a lower bound to the conductivity of G

The chromatic polynomial

We define $\chi(G,k)$ to be the number of proper k -colorings of a graph G.

- For $k \in \{0,1,2\}$ it can be computed in polynomial time (exercise).
- For $k = 3$ it is $\sharp\mathbf{P}$-complete even for bipartite graphs (Linial 1986).
- $\chi(G,\lambda)$ is a polynomial in λ (Birkhoff 1912).
- $\chi(G,-1)$ is the number of acyclic orientations of G (Stanley 1973).

The Tutte polynomial

The Tutte polynomial of G is defined as

$$T(G; X, Y) = \sum_{F \subseteq E(G)} (X-1)^{r\langle E\rangle - r\langle F\rangle} (Y-1)^{n\langle F\rangle}$$

where $k\langle F\rangle$ is the number of connected components of the spanning subgraph defined by F,

$r\langle F\rangle = |V| - k\langle F\rangle$ is its rank

and $n\langle F\rangle = |F| - |V| + k\langle F\rangle$ is its nullity.

The Tutte polynomial subsumes the chromatic polynomial.

$$\chi(G, X) = (-1)^{r(G)} \cdot X^{k(G)} \cdot T(G(1-X, 0)$$

Evaluations of the Tutte polynomial

See D. Welsh, Complexity: Knots, colourings and counting, Cambridge, 1993, and
M. Korn and I. Pak, Tilings of rectangles with T-tetrominoes, TCS 319, 2004

(i) $T(G;1,1)$ is the number of spanning trees of G,

(ii) $T(G;1,2)$ is the number of connected spanning subgraphs of G,

(iii) $T(G;2,1)$ is the number of spanning forests of G,

(iv) $T(G;2,2)$ is the number of spanning subgraphs of G,

(v) $T(G;1-k,0)$ is the number of proper k-vertex colorings of G,

(vi) $T(G;2,0)$ is the number of acyclic orientations of G,

(vii) $T(G;0,-2)$ is the number of Eulerian orientations of G.

(viii) $2 \cdot T(Grid_{4x,4y};3,3)$ is the number of tilings of the $(4x \times 4y)$- grid graph with T-tetrominoes

The cover polynomial

Chung and Graham, 1995 and D'Antona and Munarini, 2000

Let $D = (V, E)$ be a directed graph.

$C \subseteq E$ is a **path-cycle cover of G** if C is a subgraph with maximal in-degree ≤ 1 and maximal out-degree ≤ 1 and C is a vertex disjoint decomposition of E with $p(C)$ paths and $c(C)$ cycles.

The **(factorial) cover polynomial** is the polynomial

$$C(D, x, y) = \sum_{C} (x)_{p(C)} \cdot y^{c(C)}$$

The **(geometric) cover polynomial** is the polynomial

$$C_{geom}(D, x, y) = \sum_{C} (x)^{p(C)} \cdot y^{c(C)}$$

Here $(x)_n = x \cdot (x-1) \cdot \ldots \cdot (x-n+1)$ is the falling factorial.

Evaluations of the cover polynomial

(i) $C(D, 0, 1)$ is the number of cycle covers of D, which is the permanent of the adjacency matrix of D.

(ii) $C(D, 0, -1)$ is the determinant of the adjacency matrix of D.

(iii) $C(D, 1, 0)$ is the number of hamiltonian paths of D.

(iv) $C(D, x, 1)$ is the *factorial rook polynomial* of D.

Definability and Complexity

What I find intriguing?

- If one finds (in the literature) or defines (a new) a graph polynomial, how can one guess evaluations which are combinatorially meaningful?
- How are the computational difficulties of evaluating graph polynomials at different evaluation points related?

Note:

The characteristic polynomial is polynomial time computable at all evaluation points.

All other examples have many evaluation points which are $\sharp\mathbf{P}$-hard.

Evaluations, coefficients and zeros of graph polynomials

How could one prove that a graph parameter f is

not coded

in a given graph polynomial from an infinite class of graph polynomials $\mathcal{P}$ as

- an evaluation?
- a coefficient?
- a zero?

We will use method of **logic** and **linear algebra** !

Numeric graph invariants aka graph parameters

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Evaluations of graph polynomials are graph parameters,

but there are many more.

We look through our favorite monographs on graph theory:

- R. Diestel, Graph Theory, Springer 1996
- B. Bollobás, Modern Graph Theory, Springer 1998

Numeric graph invariants (graph parameters)

We denote by $G = (V(G), E(G))$ a graph,
and by $\mathcal{G}$ and $\mathcal{G}_{simple}$ the class of finite (simple) graphs, respectively.

A **numeric graph invariant** or **graph parameter** is a function

$$f : \mathcal{G} \longrightarrow \mathbb{R}$$

which is invariant under graph isomorphism.

(i) Cardinalities: $|V(G)|$, $|E(G)|$

(ii) Counting configurations:

$k(G)$ the number of connected components,
$m_k(G)$ the number of k-matchings

(iii) Size of configurations:

$\omega(G)$ the clique number
$\chi(G)$ the chromatic number

(iv) Evaluations of graph polynomials:

$\chi(G, \lambda)$, the chromatic polynomial, at $\lambda = r$ for any $r \in \mathbb{R}$.
$T(G, X, Y)$, the Tutte polynomial, at $X = x$ and $Y = y$ with $(x, y) \in \mathbb{R}^2$.

Multiplicative graph parameters

Let $G_1 \sqcup G_2$ denote the disjoint union of two graphs.

f is **multiplicative** if $f(G_1 \sqcup G_2) = f(G_1) \cdot f(G_2)$.

(i) $|V(G)|, |E(G)|, k(G)$ are not multiplicative

(ii) $\chi(G)$ and $\omega(G)$ are not multiplicative

(iii) The number of perfect matchings $pm(G)$ is multiplicative and so is the generating matching polynomial $\sum_k m_k(G)X^k$.
Note that $m_k(G)$ is not multiplicative.

(iv) The graph polynomials $\chi(G, \lambda)$ and $T(G, X, Y)$ are multiplicative.

Additive graph parameters

Let $G_1 \sqcup G_2$ denote the disjoint union of two graphs.

f is **additive** if $f(G_1 \sqcup G_2) = f(G_1) + f(G_2)$.

(i) $|V(G)|, |E(G)|$ are additive.

(ii) $k(G)$ are additive

Let $b(G)$ be the number of 2-connected components of G.
$b(G)$ is additive.

(iii) $\chi(G)$ and $\omega(G)$ are not additive

(iv) If f is additive and $r \in \mathbb{R}$, then r^f is multiplicative.

Maximizing and minimizing graph parameters

Let $G_1 \sqcup G_2$ denote the disjoint union of two graphs.

f is **maximizing** if $f(G_1 \sqcup G_2) = \max\{f(G_1), f(G_2)\}$.

f is **minimizing** if $f(G_1 \sqcup G_2) = \min\{f(G_1), f(G_2)\}$.

(i) The various chromatic numbers $\chi(G)$, $\chi_e(G)$, $\chi_t(G)$ are maximizing.

(ii) The size of the maximal clique $\omega(G)$ and the maximal degree $\Delta(G)$ are maximizing.

(iii) The tree-width $tw(G)$ and the clique-width $cw(G)$ of a graph are maximizing.

(iv) The minimum degree $\delta(G)$, the girth $g(G)$ are minimizing.

The girth is the minimum length of a cycle in G.

The connection matrix of a graph parameter

Connection matrix $M(f, 0)$.

Let G_i be an enumeration of all finite graphs (up to isomorphism).

The **connection matrix** $M(f, 0) = m_{i,j}(f, 0)$ is defined by

$$m_{i,j}(f, 0) = f(G_i \sqcup G_j)$$

The **rank** of $M(f, 0)$ is denoted by $r(f, 0)$.

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Examples: Check with $|V(G)|$ and $2^{|V(G)|}$.

Computing $r(f, 0)$

Proposition:

(i) If f is multiplicative, $r(f, 0) = 1$.

(ii) If f is additive, $r(f, 0) = 2$.

(iii) If f is maximizing or minimizing, $r(f, 0)$ is infinite.

(iv) For the average degree $d(G)$ of a graph, $r(d, 0)$ is infinite.

Proof: The first three statements are easy.

For $f = d(G)$ we have

$$M(d, 0) = 2\frac{|E_1| + |E_2|}{|V_1| + |V_2|}.$$

This contains, for graphs with a fixed number e of edges, the Cauchy matrix $(\frac{2e}{i+j})$, hence $r(d, 0)$ is infinite. $\square$.

Characterizing multiplicative graph parameters

M. Freedman, L. Lovász and A. Schrijver, 2007

Theorem: ([FLS] Proposition 2.1.)

Assume $f(G) \neq 0$ for some graph G.
f is multiplicative iff $M(f, 0)$ has rank 1 and is positive semi-definite.

Recall: A finite square matrix M over an ordered field is **positive semi-definite** if for all vectors $\bar{x}$ we have $\bar{x} M \bar{x}^{tr} \geq 0$. An infinite matrix is positive semi-definite, if every finite principal submatrix is positive semi-definite.

Definability of graph polynomials

in Monadic Second Order Logic MSOL

Simple MSOL-definable graph polynomials

The graph polynomial $ind(G,X) = \sum_i ind(G,i) \cdot X^i$, can be written also as

$$ind(G,X) = \sum_{I \subseteq V(G)} \prod_{v \in I} X$$

where I ranges over all independent sets of G.
To be an independent set is definable by a formula of Monadic Second Order Logic (MSOL) $\phi(I)$.

A **simple MSOL-definable graph polynomial** $p(G,X)$ is a polynomial of the form

$$p(G,X) = \sum_{A \subseteq V(G) : \phi(A)} \prod_{v \in A} X$$

where A ranges over all subsets of $V(G)$ satisfying $\phi(A)$ and $\phi(A)$ is a MSOL-formula.

General MSOL-definable graph polynomials

For the general case

- One allows several indeterminates $X_1, \ldots, X_t$.
- One gives an inductive definition.
- One allows an ordering of the vertices.
- One requires the definition to be **invariant under the ordering**, i.e., different orderings still give the same polynomial.
- This also allows to define the modular counting quantifiers $C_{m,q}$ "there are, modulo q exactly m elements..."

The general case includes the Tutte polynomial, the cover polynomial, and virtually all graph polynomials from the literature.

A graph polynomial which is **not MSOL-definable**

A proper coloring of a graph is **harmonious** if every pair of colors appears at most once on an edge.

Let $\chi_{harm}(G)$ be the smallest k such that G has a harmonious coloring with k colors.

Let $\chi_{harm}(G, k)$ denote the number of harmonious colorings of G with k colors.

Theorem: (Kotek, Makowsky, Zilber, 2008)

(i) $\chi_{harm}(G, k)$ is a polynomial in k

(ii) Assuming $\mathbf{P} \neq \mathbf{NP}$, $\chi_{harm}(G, k)$ is **not an MSOL-definable** polynomial in the most general sense.

A graph polynomial which is not MSOL-definable - Proofs

(i):

Verify the extension property, that adding unused colors preserves harmonicity. Then it follows from Makowsky and Zilber (2006) [MZ].

(ii):

It was shown by Edwards and McDiarmid (1995) that computing $\chi_{harm}(G)$ is **NP**-hard even on trees. It was shown by Courcelle, Makowsky, Rotics (2000) that for all $t \in \mathbb{N}$ evaluating MSOL-definable polynomials on rational points is in **P** for graphs of tree-width at most t. $\square$

Finiteness of $r(f, 0)$

Finite Rank Theorem:([GKM])

Let $p(G, \bar{X})$ be a general MSOL-definable graph polynomial
in t indeterminates, and $\bar{x} \in \mathbb{R}^t$.
Then the rank $r(p(G, \bar{x}), 0)$ is finite.

Freedman, Lovász and Schrijver introduced various connection matrices $M(f, \otimes_k)$ for k-labeld graphs and various operations, which they used to characterize graph parameters arising from partition functions and coloring models.

The theorem above holds also for the **several variations** of connection matrices for labeled graphs and relational structures.

The theorem only depends on the fact that the connection matrix $M(f, \otimes)$ is defined for an **MSOL**-smooth operation $\otimes$ in the sense of [M].

Proof: We use the **bilinear** version of the **Feferman-Vaught Theorem** for graph polynomials from [M] to estimate the $r(f, k)$.

The estimates are very bad, and just suffice to establish finiteness of $r(f, k)$,

but they grow with multiple exponentials in k.

Applications of the Finite Rank Theorem, I

Corollary:[GKM] The following numeric graph invariants f have $r(f,k) < \infty$:

(i) The number of acyclic orientations

(ii) The number of eulerian orientations

Proof: They are both instances of the Tutte polynomial.

Applications of the Finite Rank Theorem, II

Corollary:([GKM])

$\omega(G)$ **is not an instance** of any MSOL-definable graph polynomial, but **is the degree** of some MSOL-definable graph polynomial,

Proof: $\omega(G_1 \sqcup G_2) = \max\{\omega(G_1), \omega(G_2)\}$. So $r(\omega, 0) = \infty$.

$\omega(G)$ can be obtained as degree of the graph polynomial

$$clique(G, X) = \sum_i clique_i(G) X^i = \sum_{C \subseteq V} \prod_{v \in C} X$$

where $clique_i(G)$ is the number of cliques of size i,
and C varies over all cliques of G.

Clearly, $clique(G, X)$ is a simple MSOL-definable graph polynomial. $\square$

Applications of the Finite Rank Theorem, III

Corollary:([GKM])

(i) If f satisfies $f(G_1 \sqcup G_2) = \max\{f(G_1), f(G_2)\}$, then f is **not an instance** of an MSOL-definable graph polynomial.

(ii) If f satisfies $f(G_1 \sqcup G_2) = \min\{f(G_1), f(G_2)\}$, then f is **not an instance** of an MSOL-definable graph polynomial.

Applications of the Finite Rank Theorem, IV

Let $d(G)$ denote the **average degree** of G. We have

$$d(G) = \frac{1}{|V(G)|} \cdot \sum_{v \in V(G)} d_G(v),$$

where $d_G(v)$ denotes the degree of a vertex v of G.

Corollary:([GKM])

$d(G)$ is **not an instance** of an MSOL-definable graph polynomial.

Proof: For $f = d(G)$ we have

$$M(d,0) = 2\frac{|E_1| + |E_2|}{|V_1| + |V_2|}.$$

This contains, for graphs with a fixed number e of edges, the Cauchy matrix

$$(\frac{2e}{i+j}),$$

hence $r(d,0)$ is infinite. □

Specific coefficients of graph polynomials

Let $p(G, \bar{X})$ be an $MSOL$-definable graph polynomial with values in $\mathbb{R}[\bar{X}]$ with m indeterminates $X_1, \ldots, X_m$, and let

$$X_1^{\alpha_1} \cdot X_2^{\alpha_2} \cdot \ldots \cdot X_m^{\alpha_m}$$

be a **specific monomial** of $p(G, \bar{X})$ with coefficient $c_\alpha(G)$, where $\alpha = (\alpha_1, \ldots, \alpha_m)$.

Theorem:[GKM] Then there is an invariantly $MSOL$-definable graph polynomial $p_\alpha(G, \bar{X})$ such that $c_\alpha(G)$ is an evaluation of $p_\alpha(G, \bar{X})$.

Remark: The theorem remains valid for monomials of the form

$$X_1^{n_1(G)-\alpha_1} \cdot X_2^{n_2(G)-\alpha_2} \cdot \ldots \cdot X_m^{n_m(G)-\alpha_m},$$

where $n_i(G) = |V(G)|$ or $n_i(G) = |E(G)|$.

This can be used to treat the coefficient of $X^{|V(G)|-3}$ of the characteristic polynomial.

Graph polynomials which are not $\mathbf{MSOL}$-definable

without the assumption $\mathbf{P} \neq \mathbf{NP}$

Let $c : E \rightarrow [k]$ be an edge-coloring. c is **path-rainbow** if between any two vertices $u, v \in V$ there as a path where all the edges have different colors.

We denote by $\chi_{rainbow}(G, k)$ the number of path-rainbow colorings of G with k colors.

Theorem:(T. Kotek and J.A.M.)

(i) $\chi_{rainbow}(G, k)$ is a polynomial in k.

(ii) $\chi_{rainbow}(G, k)$ is not $\mathbf{MSOL}$-definable (but $\mathbf{SOL}$-definable).

Proof: A more sophisticated use of connection matrices.

The same works also for harmonious colorings.

Complexity of evaluations

References for Complexity, I

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- N. Linial,
  Hard enumeration problems in geometry and combinatorics,
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- F. Jaeger, D.L. Vertigan, D.J.A. Welsh,
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  Math. Proc. Cambridge Philos. Soc., 108 (1990) pp. 35-53.

References for Complexity, II

- Markus Bläser, Holger Dell,
  The complexity of the cover polynomial.
  ICALP'07, pp. 801-812

- Markus Bläser, Christian Hoffmann,
  On the Complexity of the Interlace Polynomial,
  STACS'08, pp. 97-108

- Markus Bläser, Holger Dell, J.A. Makowsky,
  Complexity of the Bollobas-Riordan Polynomial:
  Exceptional points and uniform reductions,
  CSR'08, pp. 86-98

The complexity of the chromatic polynomial, I

Theorem:

- $\chi(G, 3)$ is $\sharp\mathbf{P}$-complete (Valiant 1979).
- $\chi(G, -1)$ is $\sharp\mathbf{P}$-complete (Linial 1986).

Question: What is the complexity of computing $\chi(G, \lambda)$ for $\lambda = \lambda_0 \in \mathbb{Q}$ or even for $\lambda = \lambda_0 \in \mathbb{C}$?

The complexity of the chromatic polynomial, II

Let $G_1 \bowtie G_2$ denote the join of two graphs.

We observe that

$$\chi(G \bowtie E_n, \lambda) = (\lambda)^{\underline{n}} \cdot \chi(G, \lambda - n) \qquad (\star)$$

Hence we get

(i) $\chi(G \bowtie E_1, 4) = 4 \cdot \chi(G, 3)$

(ii) $\chi(G \bowtie E_n, 3 + n) = (n + 3)^{\underline{n}} \cdot \chi(G, 3)$ hence for $n \in \mathbb{N}$ with $n \geq 3$ it is $\sharp\mathbf{P}$-complete.

The complexity of the chromatic polynomial, III

If we have have an oracle for some $q \in \mathbb{Q} - \mathbb{N}$ which allows us to compute $\chi(G, q)$ we can compute $\chi(G, q')$ for any $q' \in \mathbb{Q}$ as follows:

Algorithm $A(q, q', \mid V(G) \mid)$**:**

(i) Given G the degree of $\chi(G, q)$ is at most $n = \mid V(G) \mid$.

(ii) Use the oracle and $(\star)$ to compute $n + 1$ values of $\chi(G, \lambda)$.

(iii) Using Lagrange interpolation we can compute $\chi(G, q')$ in polynomial time.

We note that this algorithm is purely algebraic and works for all graphs G, $q \in (F) - \mathbb{N}$ and $q' \in F$ for any field F extending $\mathbb{Q}$.

The complexity of the chromatic polynomial, IV

We summarize the situation for the chromatic polynomial as follows:

(i) We have an **exception set** $C = \mathbb{N}$ which is a countable union of semi-algebraic sets of dimension 0 in the field $\mathbb{C}$.

(ii) We have a numeric graph invariant $f(G) = \mid G \mid$ which is $\mathbf{FP}$.

(iii) We have **one algebraic** algorithm $A(q, q', f(G))$ which runs in polynomial time in q, q' and $f(G)$ which calls the oracle $\chi(-, q')$.
q, q' are in any finite dimensional algebraic extension field F of $\mathbb{Q}$.

(iv) The algorithm $A(q, q', f(G))$ reduces **uniformly**,
for any $q \in F - C$, the evaluation of $\chi(G, q)$ into the evaluation of $\chi(G, 3)$.

The nature of the algorithm A, I

In the case of $\chi(-, q)$ and $\chi(-, q')$

- The input of A is $f(G)) \in F$,
  in this case the degree of the $\chi(G, \lambda)$
- The output of A is a rational function $A(q, q', f(G)) \in F(x_0, x_1, \ldots x_{f(G)+2})$.
  the Lagrange interpolation for $f(G) + 1$ points for q, q'
- The final result of the reduction is obtained by evaluating this rational function at

$$x_0 = \chi(G, q'),\ x_1 = \chi(G \bowtie E_1, q'),\ \ \ldots,\ \ x_n = \chi(G \bowtie E_n, q')$$
$$x_{n+1} = q,\ x_{n+2} = q'$$

A suitable model of computation for A is

the unit-cost model BSS
advocated by L. Blum, M. Shub and S. Smale.

The uniform difficult point property for $\chi(G, \lambda)$

(i) We have shown:

For all $q \in \mathbb{Q} - \{0, 1, 2\}$ and $q' \in \mathbb{Q}$ the numeric graph invariants $\chi(-, q)$ and $\chi(-, q')$ polynomial time Turing reducible to each other.

(ii) But we have shown much more:

There is ONE algebraic reduction scheme
for all the instances $\chi(G, q)$ to $\chi(G, q')$,
where q, q' are not in $\mathbb{N}$.

Uniform algebraic reductions for evaluations of graph polynomials.

Let $f = \Phi(G, \bar{q})$ and $g = \Phi(G, \bar{q}')$ two evaluations of the same graph polynomial Φ. We say that f algebraically reduces to g uniformly in $\bar{q}, \bar{q}'$, and we write $f <^{P}_{UA} g$, if there exists

(i) a finite set $\mathcal{A}_\Phi = \{\alpha_1, \ldots, \alpha_a\}$ of size a of polynomial time computable numeric graph invariants $\alpha : Graphs \rightarrow \mathbb{Q}$, depending on Φ only;

(ii) a polynomial time computable family $r_i : i \in \mathbb{N}$ of polynomial time computable graph transductions $r_i : Graphs \rightarrow Graphs$, depending on Φ only;

The family is polynomial time computable in Φ and i.

(iii) a polynomial time computable function $A_\Phi : \mathbb{Q}^a \rightarrow \mathbb{Q}(x_1, x_2, \ldots)$, depending on Φ only;

such that for every $G \in Graphs$ we have that

$$f(G) = A_\Phi(\alpha_1(G), \ldots, \alpha_a(G)) \left(g(r_1(G), \ldots, g(r_{poly(G)}(G), \bar{q}, \bar{q}'\right)$$

The uniform difficult point property UDPP

Let $\Phi(G, \bar{x}^m)$ be a graph polynomial in m variables.

$\Phi(G, \bar{x}^m)$ has the uniform difficult point property (DPP) if the following holds:

There exists an **exception set** C_Φ which is a countable union of semi-algebraic sets of dimension $< m$ in the field $\mathbb{C}$, and for all q not in the exception set C, $\Phi(-q)$ is $\sharp\mathbf{P}$ hard.

Furthermore, for any $\bar{q}_1, \bar{q}_2 \in F^m - C_\Phi$ we have

$$\Phi(G, \bar{q}_1) <^P_{UA} \Phi(G, \bar{q}_2).$$

In other words, all the evaluations for $\bar{q}$ not in the exception set, are of the same difficulty and uniformly algebraically reducible to each other.

The Tutte polynomial

The paradigm of the DPP was inspired by the work of
Linial and Jaeger, Vertigan and Welsh.

(i) For the classical Tutte polynomial, the **uniform DPP** was proven by Jaeger, Vertigan and Welsh in 1990.

(ii) For the colored Tutte polynomial as defined by Bollobás and Riordan (1999), the **uniform DPP** was proven by Bläser, Dell and Makowsky in 2007.

(iii) This also holds for the multivariate Tutte polynomial, the Pott's model, if restricted to a fixed finite number of variables.

More polynomials with the uniform DPP

The uniform DPP was also proven for

(i) the cover polynomial $C(G, x, y)$ introduced by Chung and Graham (1995) by Bläser Dell, 2007

(ii) the interlace polynomial (aka Martin polynomial) introduced by Martin (1977) and independently by Arratia, Bollobás and Sorkin (2000), by Bläser and Hoffmann, 2007

(iii) the matching polynomial, by Averbouch, Kotek and Makowsky, 2007

(iv) the harmonious chromatic polynomial, by Averbouch, Kotek and Makowsky, 2007

What is the pattern behind this?

In establishing the UDPP one uses the fact that in the examples the evaluations at integer points are in $\sharp\mathbf{P}$.

We call such graph polynomials counting.

There seems to be dichtomy property:

- Either all the evaluations of a graph polynomial Φ are polynomial time computable, or
- Φ has the uniform difficult point property UDPP.

Conjecture: This dichtomy holds for all counting $\mathbf{MSOL}$-definable graph polynomials.

Note that it holds for the harmonious chromatic polynomial, which is not $\mathbf{MSOL}$-definable.

Thank you for your attention !