

# Set theory workshop: abstracts

## **Cardinal invariants of projections in the Calkin algebra** *Tristan Bice*

Cardinal invariants defined by combinatorial properties of  $\mathcal{P}(\omega)/\text{Fin}$ , such as the tower number, the minimum cardinality of a maximal almost disjoint family, the splitting number, etc. have been studied for quite some time. In more recent years, some set theorists have considered analogous cardinal invariants on other structures which, in the case of my research, is the collection of projections in the Calkin algebra, the  $C^*$ -algebra of bounded operators on an infinite dimensional separable Hilbert space modulo the compact operators. While similar in some ways, this latter structure has some subtle differences which make it more difficult (and interesting!) to work with. For example, it is not a lattice, although I have a nice way of determining when an arbitrary finite, or even countable, collection of projections has a l.u.b. or g.l.b. and, if not, how exactly this fails, which I will present in my speech. Also, even though projections have a unique orthogonal complement, they will also have many other complements, which often leads to each cardinal invariant having (at least) two analogs in the Calkin algebra. I will outline my recent work on the relations between these analogous cardinal invariants, as well as their relation to the more classical cardinal invariants.

## **Indestructibility of Vopenka's Principle** *Andrew Brooke-Taylor*

One method for proving that large cardinals are compatible with various other set-theoretic statements is to start with a model containing a large cardinal, and force the other statement while preserving the large cardinal property. This can often be achieved by Silver's method of lifting embeddings, in which a generic is chosen to contain a particular master condition. Laver showed that with an appropriate preparatory forcing, a supercompact cardinal can be made indestructible to a variety of forcings, in that any generic for those forcings will preserve its supercompactness; Hamkins has obtained similar results for a wide variety of large cardinals. We shall focus on the large cardinal property known as Vopenka's principle, and show that even without any preparatory forcing, it is indestructible to a broad class of forcing notions.

### **On successors of $\mathbb{R}$** *Andres Caicedo*

Assume that  $V = L(\mathcal{P}(\mathbb{R}))$  and that the strengthening of determinacy known as  $\text{AD}^+$  holds. Under this assumption we study the structure of cardinals above the size of the reals. A successor of  $\mathbb{R}$  is  $\omega_1 \cup \mathbb{R}$ . We prove a version of the  $G_0$  dichotomy of Kechris-Solecki-Todorćevic; it implies that any set is either well-orderable or else injects the reals, and that  $2^{\mathbb{N}}/E_0$  is another successor of  $\mathbb{R}$ . Woodin has identified yet another successor when  $\text{DC} + \text{AD}_{\mathbb{R}}$  holds. We identify a different successor when  $\text{AD}_{\mathbb{R}}$  fails.

### **Tukey types of ultrafilters** *Natasha Dobrinen*

This is joint work with Stevo Todorćevic

We say that a partial ordering  $P$  is *Tukey-reducible* to a partial ordering  $Q$  ( $P \leq_T Q$ ) if there is a *Tukey map*  $g : P \rightarrow Q$  such that for each unbounded subset  $X \subseteq P$ , the image  $g''X$  is unbounded in  $Q$ . Equivalently,  $P \leq_T Q$  iff there is a *cofinal map*  $f : Q \rightarrow P$  such that for each cofinal subset  $X$  of  $Q$ , the image  $f''X$  is cofinal in  $P$ .  $P \cong_T Q$  iff  $P \leq_T Q$  and  $Q \leq_T P$ .

When restricted to ultrafilters, Tukey reduction is a generalization of the well-studied Rudin-Keisler reduction between ultrafilters. Given ultrafilters  $\mathcal{U}$  and  $\mathcal{V}$  on  $\omega$ ,  $\mathcal{U} \cong_T \mathcal{V}$  iff there is a partial ordering into which  $\mathcal{U}$  and  $\mathcal{V}$  can both be cofinally embedded.

We present some results on the structure of the Tukey equivalence classes of ultrafilters on countable base sets. One of the fundamental results is that if  $\mathcal{U}$  is a p-point and  $\mathcal{U} \geq_T \mathcal{V}$ , then there is a continuous function  $f : \mathcal{P}(\omega) \rightarrow \mathcal{P}(\omega)$  such that  $f : \mathcal{U} \rightarrow \mathcal{V}$  is a monotone cofinal map. Using this, we show that many structures are embeddable into the Tukey classes of p-points and other types of ultrafilters.

### **Constructing square- $\omega_1$ by means of forcing with finite conditions** *Gregor Dolinar*

We define a forcing notion  $P$  which consists of finite conditions and generates a square sequence with domain  $\omega_2$  in the extension. The working part of a condition from  $P$  consists of a finite set of clubs - each of them in some limit ordinal  $\alpha < \omega_2$ . These clubs satisfy square-like coherency and nontriviality conditions. Each forcing condition also contains a finite set of elementary submodels of  $H_{\omega_2}$ , which follows Todorćevic's idea of using models as side conditions. We show how the aforementioned models help us establish the properness of  $P$ . Our forcing notion is inspired by Baumgartner's forcing to add a club on  $\omega_1$  using finite conditions, as well as by Abraham's rendition of that forcing.

## **Graphs on Polish spaces** *Stefan Geschke*

We investigate the structure of clopen graphs on Polish spaces.

## **Infinite games with imperfect information** *Daisuke Ikegami*

In 1953, Gale and Stewart developed the general theory of infinite games so-called Gale-Stewart games which are two-player zero-sum infinite games with perfect information. The theory of Gale-Stewart games has been deeply investigated by many logicians and now it is one of the main topics in set theory and it has connections with other topics of set theory as well as model theory and computer science. In 1969, Blackwell proved the extension of von Neumann's mini-max theorem where he introduced infinite games with imperfect information nowadays called Blackwell games. Although Blackwell games have not been so investigated as Gale-Stewart games, in 1998, Martin proved that the Axiom of Determinacy (AD) implies the Axiom of Blackwell Determinacy (BI-AD) and conjectured the converse, which is still not known to be true. In this talk, we introduce Blackwell games and their determinacy and talk about the connection between the Axiom of Real Determinacy ( $AD_{\mathbb{R}}$ ) and the Axiom of Real Blackwell Determinacy ( $BI - AD_{\mathbb{R}}$ ) where players choose real numbers instead of natural numbers.

## **Measure algebras** *Thomas Jech*

A measure algebra is a sigma-algebra that carries a strictly positive sigma-additive measure. A problem of John von Neumann from 1935 is to describe measure algebras in purely algebraic terms.

We present several algebraic characterizations of measure algebras. We shall also consider related properties such as continuous outer measures, exhaustive submeasures, sigma-finite and sigma-bounded chain conditions and distributive laws, as well as Maharam's Problem and the Control Measure Problem.

In addition we shall discuss the history of the problem and the work of Maharam, Kelley, Kalton, Roberts, Talagrand, Balcar and others.

## **Polarized Partition Properties on the Second Level of the Projective Hierarchy** *Yurii Khomskii*

Polarized partition properties for sets of real numbers are variants of the classical Ramsey property which have come under attention recently, in the work of DiPrisco, Todorcevic, Zapletal and Shelah among others. Just like other regularity properties,

they are true of all Borel and analytic sets, but the question whether this can be extended to the next level, i.e., the  $\Delta_2^1$  and  $\Sigma_2^1$  sets, is independent of ZFC. We investigate the logical strength of the statements “the polarized partition property holds for all  $\Delta_2^1/\Sigma_2^1$  sets” and compare it to other well-known regularity properties on the second level of the projective hierarchy.

### **Descriptive set theory of CLI-groups** *Maciej Malicki*

The classical theorem of Birkhoff-Kakutani implies that every metrizable topological group  $G$  admits a compatible left-invariant metric, that is, a metric  $d$  inducing the group’s topology, and such that  $d(zx, zy) = d(x, y)$  for all  $x, y, z$  in  $G$ . However, even if  $G$  is completely metrizable, this theorem says nothing about the existence of a metric that is at the same time left-invariant and complete. Actually, there exist many natural examples of completely metrizable groups with no such metric.

Polish groups admitting a complete left-invariant metric, called CLI groups, arise naturally in a number of problems. H.Becker proved, among other things, that the Glimm-Effros dichotomy holds for orbit equivalence relations of actions associated with CLI groups. Alex Thompson proved a partial converse to an important theorem of Kechris. Kechris’ result says that every orbit equivalence relation associated with a continuous action of a Polish locally compact group is essentially countable. Thompson showed that every non-CLI group has an action whose orbit equivalence relation is not essentially countable.

In this talk, we investigate the family of all CLI groups from a point of view of descriptive set theory. However, it surprisingly turns out that a descriptive-set theoretic result leads to a general and purely topological-group theoretic statement. The first main theorem discussed in the talk states that the family of all CLI groups is coanalytic non-Borel, regarded as a subset of a naturally defined standard Borel space of all Polish groups. Then this theorem is used to show that there does not exist a universal CLI group, nor a surjectively universal CLI group (that is, there does not exist a CLI group containing an isomorphic copy of every CLI group, nor a CLI group which can be homomorphically mapped onto every CLI group.) This answers a question posed by H.Becker.

### **Infinitary Logic and Large Cardinals** *Denis I. Saveliev*

I plan to talk on infinitary logics  $\mathcal{L}_{\kappa, \lambda}$  and some their extensions, with particular emphasis on their completeness. A general discussed problem is: Given a theorem of the usual finitary logic  $\mathcal{L}_{\omega, \omega}$ , as Löwenheim–Skolem, Compactness, Completeness,

Ultraproduct, etc., for which  $\kappa, \lambda$  the logic  $\mathcal{L}_{\kappa, \lambda}$  satisfies an appropriate analog of that theorem? This leads to large cardinals. A part of the talk will be devoted to basic facts of infinitary logic and theory of large cardinals, and another part to my recent results on the Infinitary Completeness Theorem. I show that in fact Completeness is equivalent to Compactness; any consistent theory in  $\mathcal{L}_{\kappa, \lambda}$  has a model if and only if  $\kappa$  is strongly compact, and any consistent theory in  $\mathcal{L}_{\kappa, \lambda}$  using at most  $\kappa$  non-logical symbols has a model if and only if  $\kappa$  is weakly compact. Likewise for higher-order infinitary logics; any consistent theory in  $\mathcal{L}_{\kappa, \lambda}^n$  has a model if and only if  $\kappa$  is extendible. I discuss also infinitary analogs of several other logical theorems

**Some parametrization theorems for measurable sets with uncountable sections**  
*Shashi Mohan Shrivastava*

In this note we prove the following result. This improves the results of Wesley and Cenzer and Maudlin.

**Theorem** *Let  $X$  be an analytic space,  $Y$  a Polish space, and  $A \subset X \times Y$  an analytic set with sections  $A_x$  uncountable. Then there is a  $B \subset A$  with sections  $B_x$  a homeomorph of the Cantor set and  $B \in \mathcal{S}_X \otimes \mathcal{B}_Y$ , where  $\mathcal{S}_X$  denotes the smallest  $\sigma$ -algebra on  $X$  containing all Borel sets and closed under Souslin operations and  $\mathcal{B}_Y$  the Borel  $\sigma$ -algebra of  $Y$ .*

**Non-structure in homogeneous model theory and constructibility relative to**  
*0# Agatha Walczak-Typke*

Structure/non-structure results in model theory are a way of finely classifying the complexity of a given type of theory. I will discuss some recent strong non-structure results in homogeneous model theory, a generalization of first-order model theory. I will then compare these results to set-theoretic notions of complexity developed by Sy Friedman. The work presented is joint with Tapani Hyttinen and Sy-David Friedman.