Model Theory of Modal Logic
Lecture 1: A brief introduction to modal logic

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Some course readings

Recommended:

1. Valentin Goranko and Martin Otto: Model Theory of Modal Logic, Chapter in: Handbook of Modal Logic, P. Blackburn, J. van Benthem, F. Wolter (eds.), Kluwer, 2007, pp. 249-329.

2. Patrick Blackburn, Maarten de Rijke, and Yde Venema: Modal Logic, Cambridge University Press, 2002.

Supplementary:

3. Valentin Goranko and Dimiter Vakarelov: Elementary Canonical Formulae: Extending Sahlqvist Theorem, Annals of Pure and Applied Logics, 2006, vol. 141, 1-2, pp. 180-217.

4. Willem Conradie, Valentin Goranko and Dimiter Vakarelov: Algorithmic correspondence and completeness in modal logic. I. The core algorithm SQEMA, Logical Methods in Computer Science, vol. 2 (1:5) 2006, pp.1-26.

Modal logic: some historical remarks

- Aristotle: the 'Sea-battle tomorrow' argument.
  Necessary truths. Future truths.
- Medieval (modal) logic: mostly about theological issues.
- Leibniz: *A is necessarily true if it is true in all possible worlds.*
- C.I. Lewis: problems with the classical ('material') implication:
  - Irrelevance/non-causality: *If the Sun is hot, then 2+2=4.*
  - Ex falsum quodlibet:
    If 2+2=5 then the Moon is made of cheese.
  - Monotonicity:
    If I put sugar in my tea, then it will taste good.

    If I put sugar and I put petrol in my tea then it will taste good.
- Lewis' proposal: to introduce a strong implication

$$A \Rightarrow B := \Box(A \to B),$$

where $\Box X$ means 'X is necessarily true'.

The emergence of modern modal logic

- Until the late 1950s: a collection of syntactic theories.
- The beginning of modern modal logic: in early 1960s with the introduction of the relational semantics by Saul Kripke.
- The philosophical idea behind the Kripke semantics is Leibniz' definition of necessary truth.
- Vigorous development of formal modal logic since the 1960s. A wide variety of modal systems, with different interpretations of the modal operators emerge.
- Gradually, modal logic changes focus and becomes increasingly popular as a versatile, suitably expressive, and computationally well-behaved framework for logical specification and reasoning in various areas of CS and AI.

Modes of truth.
Variety of modal reasoning and logics.

- Necessary and possible truths. Alletic logics.
- Truths over time. Temporal reasoning. Temporal logics.
- Reasoning about spatial relations. Spatial logics.
- Reasoning about ontologies. Description logics.
- Reasoning about knowledge. Epistemic logics.
- Reasoning about beliefs. Doxastic logics.
- Reasoning about obligations and permissions. Deontic logics.
- Reasoning about program executions. Logics of programs.
- Specification of transition systems. Logics of computations.
- Reasoning about many agents and their knowledge, beliefs, goals, actions, strategies, etc. Logics of multiagent systems.

The basic propositional modal logic ML: syntax

Language of ML: logical connectives $\bot, \neg, \wedge$, and a unary modal operator $\Diamond$, and a set of atomic propositions $\mathrm{AP} = \{p_0, p_1, ...\}$.

Formulae:

$$\varphi = p \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi$$

Definable propositional connectives:

$\top := \neg\bot;$
$\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi);$
$\varphi \to \psi := \neg(\varphi \wedge \neg\psi);$
$\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi).$

The dual operator of $\Diamond$: $\Box\varphi = \neg\Diamond\neg\varphi$.

Meanings of the modal operators

- In aletic logic:
  $\Box\varphi$: 'φ is necessarily true'; $\Diamond\varphi$: 'φ is possibly true';
- In deontic logic:
  $\Box\varphi$: 'φ is obligatory'; $\Diamond\varphi$: 'φ is permitted';
- In logic of beliefs: $\Box\varphi$: 'the agent believes φ';
  $\Diamond\varphi$: 'the agent does not disbelieve φ';
- In logic of knowledge: $\Box\varphi$: 'the agent knows that φ';
  $\Diamond\varphi$: 'φ is consistent with the agent's knowledge';
- In temporal logic: $\Box\varphi$: 'φ will always be true',
  $\Diamond\varphi$: 'φ will become true sometime in the future ',
- In logic of (non-deterministic) programs:
  $\Box\varphi$: 'φ will be true after every execution of the program',
  $\Diamond\varphi$: 'φ will be true after some execution of the program'.
- In logic of topological spaces: '$\Box\varphi$ is true at w iff w is in the interior of (the extension of) φ'; '$\Diamond\varphi$ is true at w iff w is in the closure of (the extension of) φ'.

Some important modal formulae

- T: $\Box p \rightarrow p$;
- D: $\Box p \rightarrow \Diamond p$;
- B: $p \rightarrow \Box\Diamond p$;
- 4: $\Box p \rightarrow \Box\Box p$;
- 5: $\Diamond p \rightarrow \Box\Diamond p$;
- K: $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$;

Exercise: think which of these formulae should be accepted as valid principles for each of the various meanings of the modal operators.

Semantic structures for modal logic: Kripke frames

- Kripke frame: a pair $\mathfrak{F} = (W, R)$, where:
    - $W = \mathrm{dom}(\mathfrak{F})$ is a non-empty set of possible worlds,
    - $R \subseteq W^2$ is an accessibility relation between possible worlds.

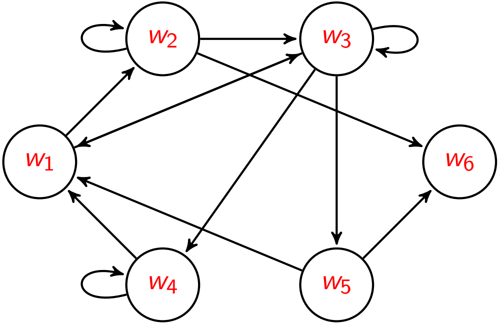
Thus, a Kripke frame is a directed graph, possibly with loops.

Depending on the context, the elements of W, are also called states, points, etc.

We will denote $R(w) := \{u \in W \mid Rwu\}$.

A pointed frame is a pair $(\mathfrak{F}, w)$ where $w \in \mathrm{dom}(\mathfrak{F})$.

Kripke frame: example



Kripke structures

A Kripke structure (Kripke model) over a Kripke frame $\mathfrak{F} = \langle W, R \rangle$ is a pair

$$\mathfrak{M} = \langle \mathfrak{F}, V \rangle$$

where $V : \mathrm{AP} \rightarrow \mathcal{P}(W)$ is a valuation, assigning to every atomic proposition p the set of states in W where p is declared true.

The set W is the domain of $\mathfrak{M}$, denoted $\mathrm{dom}(\mathfrak{M})$.

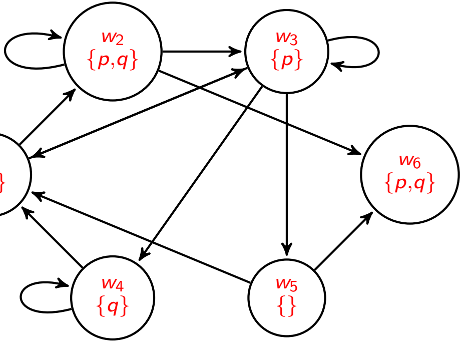
We often specify Kripke structures directly:

$$\mathfrak{M} = \langle W, R, V \rangle$$

Sometimes, instead of using valuation, a Kripke structure is specified by a labelling function $L : W \rightarrow \mathcal{P}(\mathrm{AP})$, where $L(w)$ comprises the atomic propositions true at the possible world w.

A pointed Kripke structure is a pair $(\mathfrak{M}, w)$ where $w \in \mathrm{dom}(\mathfrak{M})$.

Kripke structure: example



The valuation:
$V(p) = \{w_2, w_3, w_6\}, \quad V(q) = \{w_1, w_2, w_4, w_6\}.$

Kripke semantics of modal logic

Truth of a formula φ at a possible world u in a Kripke model $\mathfrak{M} = (W, R, V)$, denoted $\mathfrak{M}, u \models \varphi$, is defined as follows:

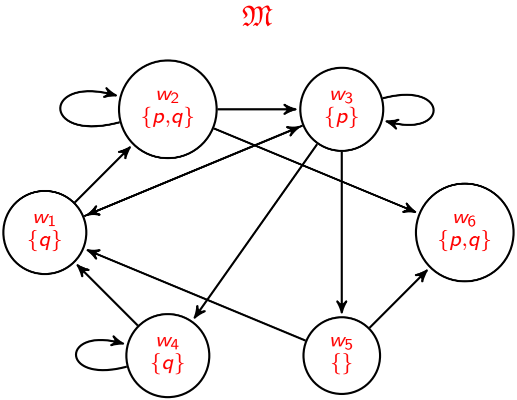
- $\mathfrak{M}, u \models p$ iff $u \in V(p)$;
- $\mathfrak{M}, u \not\models \bot$;
- $\mathfrak{M}, u \models \neg\varphi$ iff $\mathfrak{M}, u \not\models \varphi$;
- $\mathfrak{M}, u \models \varphi_1 \wedge \varphi_2$ iff $\mathfrak{M}, u \models \varphi_1$ and $\mathfrak{M}, u \models \varphi_2$;
- $\mathfrak{M}, u \models \Diamond\varphi$ iff $\mathfrak{M}, w \models \varphi$ for some $w \in W$ such that Ruw.

Respectively,
$\mathfrak{M}, u \models \Box\varphi$ if $\mathfrak{M}, w \models \varphi$ for every $w \in W$ such that Ruw.

An important feature of modal logic: the notion of truth is local, i.e., at a state of a model.

However, modal formulae cannot refer explicitly to possible worlds.

Truth of modal formulae: exercises



Check the following:

$\mathfrak{M}, w_1 \overset{?}{\models} \Box p$. Yes.

$\mathfrak{M}, w_1 \overset{?}{\models} q \land \Box q$. No.

$\mathfrak{M}, w_1 \overset{?}{\models} \Box\Diamond q$. Yes.

$\mathfrak{M}, w_2 \overset{?}{\models} \Diamond(q \land \Box q)$.
Yes.

$\mathfrak{M}, w_2 \overset{?}{\models} \Box\Box(p \lor q)$.
No.

$\mathfrak{M}, w_3 \overset{?}{\models} \Box(\neg q \to \Diamond\neg p)$. Yes.

Extension of a formula

The extension of a formula φ in a Kripke model $\mathfrak{M} = (W, R, V)$ is the set of states in $\mathfrak{M}$ satisfying the formula:

$$\|\varphi\|_{\mathfrak{M}} := \{w \mid \mathfrak{M}, w \models \varphi\}.$$

The extension of a formula $\|\varphi\|_{\mathfrak{M}}$ can be computed inductively on the construction of φ:

- $\|\bot\|_{\mathfrak{M}} = \emptyset$;
- $\|p\|_{\mathfrak{M}} = V(p)$
- $\|\neg\varphi\|_{\mathfrak{M}} = W \setminus \|\varphi\|_{\mathfrak{M}}$;
- $\|\varphi_1 \wedge \varphi_2\|_{\mathfrak{M}} = \|\varphi_1\|_{\mathfrak{M}} \cap \|\varphi_2\|_{\mathfrak{M}}$;
- $\|\Diamond\varphi\|_{\mathfrak{M}} = \{w \mid R(w) \cap \|\varphi\|_{\mathfrak{M}} \neq \emptyset\}$.

The respective clause for $\Box$: $\|\Box\varphi\|_{\mathfrak{M}} = \{w \mid R(w) \subseteq \|\varphi\|_{\mathfrak{M}}\}$.

Model checking of modal formulae

Model checking is a procedure checking whether a given model satisfies given property, usually specified in some logical language.

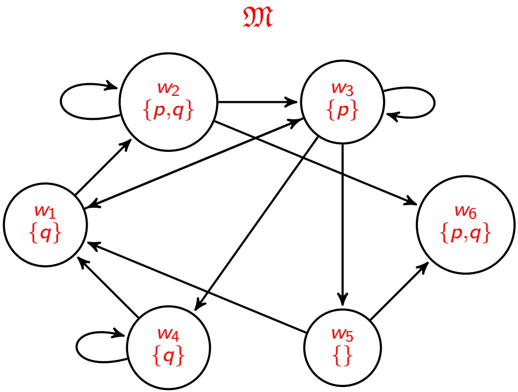
Model checking may, or may not, be algorithmically decidable, depending on the logical formalism and the class of models under consideration.

The main model checking problems for modal logic are:

1. Local model checking: given a Kripke model $\mathfrak{M}$, a state $u \in \mathfrak{M}$ and a modal formula φ, determine whether $\mathfrak{M}, u \models \varphi$;
2. Global model checking: given a Kripke model $\mathfrak{M}$ and a modal formula φ, determine the set $\|\varphi\|_{\mathfrak{M}}$.

We are also interested in model satisfiability checking: given a Kripke model $\mathfrak{M}$ and a formula φ, determine whether $\|\varphi\|_{\mathfrak{M}} \neq \emptyset$.

Global model checking of modal formulae: exercises



Compute the following:

$\|\Box p\|_{\mathfrak{M}} = \{w_1, w_2, w_6\}$.

$\|p \wedge \Box p\|_{\mathfrak{M}} = \{w_2, w_6\}$.

$\|\Diamond(p \wedge \Box p)\|_{\mathfrak{M}} = \{w_1, w_2, w_5\}$.

$\|\neg q \rightarrow \Diamond(p \wedge \Box p)\|_{\mathfrak{M}} = \{w_1, w_2, w_4, w_5, w_6\}$.

$\|\Box\Box(\neg p \rightarrow q)\|_{\mathfrak{M}} =$?

Global model checking of modal formulae: algorithm

Global model checking of modal formulae: given a (finite) Kripke model $\mathfrak{M}$ and a formula θ, compute the extensions $\|\varphi\|_{\mathfrak{M}}$ for all subformulae φ of θ recursively, by labelling all possible worlds with those subformulae of θ that are true at those worlds, as follows:

- The labelling of atomic propositions is given by the valuation.
- The propositional cases are routine.
- $\|\Diamond\varphi\|_{\mathfrak{M}}$ consists of all states which have a successor in $\|\varphi\|_{\mathfrak{M}}$, i.e., labelled by φ.

QUESTION: What is the worst case complexity of global model checking of a modal formula φ is a given finite Kripke model $\mathfrak{M}$

- in terms of the length of the formula $|\varphi|$?
- in terms of the size of the model $|\mathfrak{M}|$?

Propositional multi-modal logic

Modal language: $\mathrm{ML}(\tau, \mathrm{AP})$, where:

- τ is a set of modalities $\alpha \in \tau$.

  Each $\alpha \in \tau$ labels a modal diamond operator $\langle\alpha\rangle$.
- AP is a (countable) set of propositional variables or atomic propositions.

The formulae of $\mathrm{ML}(\tau, \mathrm{AP})$ are recursively defined as follows:

$$\phi := \bot \mid p \mid \phi_1 \rightarrow \phi_2 \mid \langle\alpha\rangle\phi,$$

where $p \in \mathrm{AP}$ and $\alpha \in \tau$.

Constant formula: formula not containing atomic propositions.

The logical constant $\top$ and connectives $\neg, \wedge, \vee, \leftrightarrow$ are defined as standard abbreviations.

Box operator dual to $\langle\alpha\rangle$: $[\alpha]$, defined by $[\alpha]\,\phi := \neg\langle\alpha\rangle\neg\phi$.

Multi-modal Kripke frames and structures

Let $\mathrm{ML}(\tau, \mathrm{AP})$ be a multi-modal language.
A (Kripke) τ-frame is a relational τ-structure

$$\mathfrak{F} = \langle W, \{R_\alpha\}_{\alpha \in \tau} \rangle$$

where:

- $W \neq \emptyset$;
- $R_\alpha \subseteq W \times W$ for each $\alpha \in \tau$.

Kripke τ-structure: $\mathfrak{M} = \langle \mathfrak{F}, V \rangle = \langle W, \{R_\alpha\}_{\alpha \in \tau}, V \rangle$.

Modal operators in Kripke frames

Given a τ-frame $\mathfrak{F} = \langle W, \{R_\alpha\}_{\alpha\in\tau}\rangle$, every R_α defines two unary operators, $\langle R_\alpha\rangle$ and its dual $[R_\alpha]$, on $\mathcal{P}(W)$ as follows:

$$\langle R_\alpha\rangle (X) := \{w \in W \mid wR_\alpha u \text{ for some } u \in X\},$$

and

$$[R_\alpha](X) := \overline{\langle R_\alpha\rangle(\overline{X})} = \{w \in W \mid wR_\alpha u \text{ for all } u \in X\},$$

where $\overline{A} := W \setminus A$ (the complement of A in W).

Extending valuations to all modal formulae

In any Kripke structure $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ the valuation V can be extended over the set of all formulae, so that

$$V(\phi) = \{w \in \mathfrak{M} \mid \mathfrak{M}, w \models \phi\}.$$

The extension of V is defined recursively as follows:

$V(\bot) := \emptyset$

$V(\phi_1 \to \phi_2) := \overline{V(\phi_1)} \cup V(\phi_2)$

$V(\langle \alpha \rangle \phi) := \langle R_\alpha \rangle (V(\phi))$

Thus, for every formulae φ, $V(\varphi) = \|\phi\|_{\mathfrak{M}}$.

Exercise: : Determine $V(\neg\phi)$, $V(\phi_1 \vee \phi_2)$, $V(\phi_1 \wedge \phi_2)$, and $V([\alpha]\, \phi)$

Remark: In algebraic terms the extended valuation is the unique homomorphism from the free τ-algebra of formulae to the modal algebra associated with the model $\mathfrak{M}$, extending V.

Validity and satisfiability of modal formulae

A τ-formula ϕ is:

(i) true at the state w of the τ-structure $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$, denoted $\mathfrak{M}, w \models \phi$, if $w \in V(\phi)$.

We also say that ϕ is true in the pointed structure $(\mathfrak{M}, w)$.

A formula, true at a state of some τ-structure is satisfiable.

(ii) valid in $\mathfrak{M}$, denoted $\mathfrak{M} \models \phi$, if $\mathfrak{M}, w \models \phi$ for every $w \in \mathrm{dom}(\mathfrak{F})$, i.e., if $V(\phi) = \mathrm{dom}(\mathfrak{F})$.

(iii) (locally) valid at the state w of $\mathfrak{F}$, denoted $\mathfrak{F}, w \models \phi$, if $\mathfrak{M}, w \models \phi$ for every τ-structure $\mathfrak{M}$ over $\mathfrak{F}$.

Then, we also say that ϕ is valid in the pointed frame $(\mathfrak{F}, w)$.

(iv) valid in $\mathfrak{F}$, denoted $\mathfrak{F} \models \phi$, if $\mathfrak{F}, w \models \phi$ for every $w \in \mathrm{dom}(\mathfrak{F})$.

Equivalently, $\mathfrak{F} \models \phi$ if $\mathfrak{M} \models \phi$ for every τ-structure $\mathfrak{M}$ over $\mathfrak{F}$.

(v) valid, denoted $\models \phi$, if $\mathfrak{F} \models \phi$ for every τ-frame $\mathfrak{F}$.

Relational languages associated with a modal language

With the modal language $\mathrm{ML}(\tau, \mathrm{AP})$, we associate the following purely relational signatures:

(i) the relational version of τ, containing $=$ and a family of binary relational symbols R_α for $\alpha \in \tau$, again denoted by τ.

(ii) the expansion τ_{AP} of the relational signatures τ by unary predicates $\{P_0, P_1, \dots\}$ associated with the atomic propositions $p_0, p_1, \dots \in \mathrm{AP}$.

$\mathrm{FO}(\tau)$ and $\mathrm{FO}(\tau_{\mathrm{AP}})$ are the first-order languages with signatures τ and τ_{AP}, respectively.

We regard τ-frames as τ-structures, and Kripke structures over τ-frames as τ_{AP}-structures, with P_i interpreted as $V(p_i)$.

Wherever necessary, we will highlight the distinction by writing $\models_{\mathrm{FO}}$ to explicitly refer to first-order semantics.

The standard translation of modal logic into FOL

Let $\mathrm{VAR} = \{x_0, x_1, \ldots\}$ be the set of individual variables of $\mathrm{FO}(\tau_{\mathrm{AP}})$. The formulae of $\mathrm{ML}(\tau)$ are translated into $\mathrm{FO}(\tau_{\mathrm{AP}})$ by means of the following standard translation, parameterised with the variables from VAR:

- $\mathrm{ST}(p_i; x) := P_i x$ for every $p_i \in \mathrm{AP}$.
- $\mathrm{ST}(\bot; x) := \bot$.
- $\mathrm{ST}(\phi_1 \to \phi_2; x) := \mathrm{ST}(\phi_1; x) \to \mathrm{ST}(\phi_2; x)$.
- $\mathrm{ST}(\langle\alpha\rangle\, \phi; x) := \exists y (x R_\alpha y \wedge \mathrm{ST}(\phi; y))$, where y is the first variable in $\mathrm{VAR} \setminus \{x\}$.

Exercise: : show that:

$\mathrm{ST}(\neg\phi; x) \equiv \neg\mathrm{ST}(\phi; x)$,

$\mathrm{ST}(\phi_1 \wedge \phi_2; x) \equiv \mathrm{ST}(\phi_1; x) \wedge \mathrm{ST}(\phi_2; x)$,

$\mathrm{ST}(\phi_1 \vee \phi_2; x) \equiv \mathrm{ST}(\phi_1; x) \vee \mathrm{ST}(\phi_2; x)$,

$\mathrm{ST}([\alpha]\, \phi; x) := \forall y (x R_\alpha y \to \mathrm{ST}(\phi; y))$.

The standard translation of modal logic into FO^2

Note that only the variable x is free in $\mathrm{ST}(\phi; x)$.

Moreover, for the standard translation of any modal formula it suffices to re-use, alternatively, only two variables, x and y.

In particular:

$$\mathrm{ST}(\langle\alpha\rangle\,\phi; y) := \exists x (y R_\alpha x \wedge \mathrm{ST}(\phi; x)).$$

This yields a translation of modal logic into the two-variable fragment FO^2 of first-order logic.

Remark: the standard translation of any modal formula falls into the guarded fragment of first-order logic.

Standard translation in basic modal logic: some examples

L_0: the FO language with $=$, a binary predicate R, and individual variables $\mathrm{VAR} = \{x_0, x_1, ...\}$.

L_1: the FO language extending L_0 with a set of unary predicates $\{P_0, P_1, ...\}$, corresponding to the atomic propositions $p_0, p_1, ...$.

Some examples of standard translations of modal formulae into L_1:

1. $\mathrm{ST}(\Box p \to p; x) = \forall y(Rxy \to Py) \to Px$.
2. $\mathrm{ST}(\Box p \to \Box\Box p; x) =$ $\forall y(Rxy \to Py) \to \forall z(Rxz \to \forall u(Rzu \to Pu))$.
3. $\mathrm{ST}(\Box\Box\Diamond p; x) = \forall y(Rxy \to \forall z(Ryz \to \exists u(Rzu \wedge Pu)))$ $\equiv \forall y(Rxy \to \forall x(Ryx \to \exists y(Rxy \wedge Py)))$.
4. $\mathrm{ST}(\Box\Diamond p \to \Diamond\Box\neg p) =$ $\forall y(Rxy \to \exists z(Ryz \wedge Pz)) \to \exists u(Rxu \wedge \forall v(Ruv \to \neg Pv))$ $\equiv \forall y(Rxy \to \exists x(Ryx \wedge Px)) \to \exists y(Rxy \wedge \forall x(Ryx \to \neg Px))$.
5. $\mathrm{ST}(\neg\Diamond p \wedge \Box(\neg q \vee \Box\Diamond\neg p); x) =?$

The semantic effect of the standard translation in Kripke structures

Proposition

For every pointed Kripke structure $(\mathfrak{M}, w)$ *and* $\phi \in \mathrm{ML}$*:*

$$\mathfrak{M}, w \models \phi \ \textit{ iff } \ \mathfrak{M}, w \models_{FO} ST(\phi; x)[x := w].$$

Consequently, for every Kripke structure $\mathfrak{M}$*:*

$$\mathfrak{M} \models \phi \ \textit{ iff } \ \mathfrak{M} \models_{FO} \forall x ST(\phi; x).$$

The semantic effect of the standard translation in Kripke frames

Proposition

For every pointed Kripke frame $(\mathfrak{F}, w)$ *and* $\phi \in \mathrm{ML}$ *with atomic propositions among* $p_0, \ldots, p_n$*:*

$$\mathfrak{F}, w \models \phi \ \text{ iff } \ \mathfrak{F}, w \models \forall P_0 \ldots \forall P_n ST(\phi; x)[x := w].$$

Consequently, for every Kripke frame $\mathfrak{F}$*:*

$$\mathfrak{F} \models \phi \text{ iff } \mathfrak{F} \models \forall P_0 \ldots \forall P_n \forall x ST(\phi; x).$$

Thus: in terms of truth and validity in Kripke structures, ML is a fragment of the first-order language L_1, while
in terms of validity in Kripke frames, it is a fragment of universal monadic second order logic over L_0.