Model Theory of Modal Logic
Lecture 2

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Classes of structures defined by a modal formula

We fix an arbitrary modal language $\mathrm{ML} = \mathrm{ML}(\tau)$.

Given a formula $\phi \in \mathrm{ML}$, we define:

$$\mathrm{PKS}(\phi) := \{(\mathfrak{M}, w) \mid \mathfrak{M}, w \models \phi\},$$

$$\mathrm{PFR}(\phi) := \Big\{(\mathfrak{F}, w) \;\Big|\; (\mathfrak{F}, w) \models \phi\Big\},$$

$$\mathrm{KS}(\phi) := \{\mathfrak{M} \mid \mathfrak{M} \models \phi\},$$

$$\mathrm{FR}(\phi) := \Big\{\mathfrak{F} \;\Big|\; \mathfrak{F} \models \phi\Big\}.$$

A class $\mathcal{P}$ of pointed Kripke structures is (modally) definable in the language ML if $\mathcal{P} = \mathrm{PKS}(\phi)$ for some formula $\phi \in \mathrm{ML}$.

Definable classes of Kripke structures, frames, and pointed frames are defined likewise.

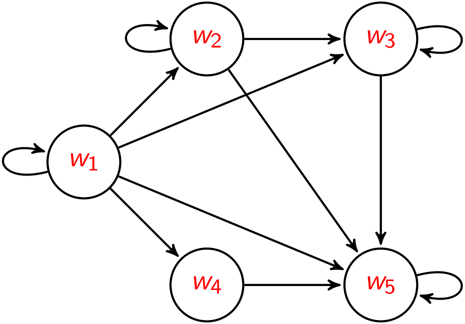
Examples of modally definable classes of Kripke structures

- The class of pointed Kripke structures $(\mathfrak{M}, w)$, where $\mathfrak{M} = \langle W, R, V \rangle$, such that w has at least one successor not satisfying p and every successor of which satisfies q, is defined by the formula $\Diamond(\neg p \wedge \Box q)$.
- The formula $\Diamond p \wedge \Diamond \neg p$ defines the class of Kripke structures in which every state has a successor satisfying p and a successor satisfying $\neg p$.
- The formula $p \rightarrow q$ defines the class of Kripke structures in which the valuation of p is included in the valuation of q.
- The formula $p \rightarrow \Box p$ defines the class of Kripke structures in which the valuation of p is closed under the accessibility relation.

Validity of modal formulae in Kripke frames

Checking validity of a modal formula φ in a frame $\mathfrak{F}$ requires checking validity of φ in all Kripke models based on $\mathfrak{F}$, i.e., for all possible valuations of the atomic propositions occurring in φ.

$\mathfrak{F}$:



Check the following:

$\mathfrak{F}, w_1 \overset{?}{\models} \Box p \to p$. Yes.

$\mathfrak{F}, w_1 \overset{?}{\models} p \to \Box\Diamond p$. No.

$\mathfrak{F}, w_2 \overset{?}{\models} \Box p \to \Box\Box p$. Yes.

$\mathfrak{F}, w_2 \overset{?}{\models} \Diamond\Diamond p \to \Diamond p$. Yes.

$\mathfrak{M}, w_1 \overset{?}{\models} \Box(\Box p \to p)$. No.

$\mathfrak{F} \overset{?}{\models} \Box(p \to q) \to (\Box p \to \Box q)$. Yes.

$\mathfrak{F} \overset{?}{\models} \Box p \to p$. No. $\mathfrak{F} \overset{?}{\models} \Box p \to \Box\Box p$. Yes.

Some properties of binary relations

A binary relation $R \subseteq X^2$ is called:

- reflexive if it satisfies $\forall x\ xRx$.
- irreflexive if it satisfies $\forall x\ \neg xRx$.
- serial if it satisfies $\forall x \exists y\ xRy$.
- functional if it satisfies $\forall x \exists! y\ xRy$,
- symmetric if it satisfies $\forall x \forall y (xRy \rightarrow yRx)$.
- asymmetric if it satisfies $\forall x \forall y (xRy \rightarrow \neg yRx)$.
- antisymmetric if it satisfies $\forall x \forall y (xRy \wedge yRx \rightarrow x = y)$.
- connected if it satisfies $\forall x \forall y (xRy \vee yRx)$.
- transitive if it satisfies $\forall x \forall y \forall z ((xRy \wedge yRz) \rightarrow xRz)$.
- equivalence relation if it is reflexive, symmetric, and transitive.
- euclidean if it satisfies $\forall x \forall y \forall z ((xRy \wedge xRz) \rightarrow yRz)$.
- pre-order, (or quasi-order) if it is reflexive and transitive.
- partial order, if it is reflexive, transitive, and antisymmetric.
- linear order, (or total order) if it is a connected partial order.
- well-founded order, if it is a partial order with no infinite strictly decreasing chains.

Some relational properties of Kripke frames definable by modal formulae

Claim For every Kripke frame $\mathfrak{F} = (W, R)$ the following holds:

- $\mathfrak{F} \models \Box p \rightarrow p$ iff the relation R is reflexive.
  Thus, the formula $\Box p \rightarrow p$ defines the class of reflexive frames.
- $\mathfrak{F} \models \Box p \rightarrow \Diamond p$ iff the relation R is serial.

  Exercise: Find a simpler modal formula that defines seriality.
- $\mathfrak{F} \models \Box p \leftrightarrow \Diamond p$ iff the relation R is a function.
- $\mathfrak{F} \models p \rightarrow \Box\Diamond p$ iff $\mathfrak{F} \models \Diamond\Box p \rightarrow p$ iff
  the relation R is symmetric.
- $\mathfrak{F} \models \Box p \rightarrow \Box\Box p$ iff $\mathfrak{F} \models \Diamond\Diamond p \rightarrow \Diamond p$ iff
  the relation R is transitive.
- $\mathfrak{F} \models \Diamond p \rightarrow \Box\Diamond p$ iff $\mathfrak{F} \models \Diamond\Box p \rightarrow \Box p$ iff
  the relation R is euclidean.

More relational properties of Kripke frames definable by modal formulae

- A challenge: $\mathfrak{F} \models \Diamond\Box p \to \Box\Diamond p$ iff …?
- A bigger challenge: $\mathfrak{F} \models \Box\Diamond p \to \Diamond\Box p$ iff …?
- An even bigger challenge: $\mathfrak{F} \models \Box(\Box p \to p) \to \Box p$ iff …?

Validity of modal formulae

Some valid modal formulae:

- Every modal instance of a propositional tautology, i.e., every formula obtained by uniform substitution of modal formulae for propositional variables in a propositional tautology.

  For instance: $\Box p \vee \neg \Box p$; $(\Box p \wedge \Diamond \Box q) \rightarrow \Diamond \Box q$, etc.

- K: $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$;

- $\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$.

- $\Diamond(p \vee q) \leftrightarrow (\Diamond p \vee \Diamond q)$.

- $\Box\varphi$, for every valid modal formula φ.

  E.g., $\Box(\Diamond p \vee \neg \Diamond p)$.

Unlike first-order logic, testing validity in modal logic is decidable, and PSPACE-complete.

Modal theories

The modal theory of a pointed Kripke τ-structure $(\mathfrak{M}, w)$ is the set of all formulae of $\mathrm{ML}(\tau)$ satisfied in $(\mathfrak{M}, w)$:

$$\mathrm{Th}_{\mathrm{ML}}(\mathfrak{M}, w) := \{\varphi \in \mathrm{ML}(\tau) \mid \mathfrak{M}, w \models \varphi\}.$$

Respectively, the modal theory of $\mathfrak{M}$ is

$$\mathrm{Th}_{\mathrm{ML}}(\mathfrak{M}) := \{\varphi \in \mathrm{ML}(\tau) \mid \mathfrak{M} \models \varphi\}.$$

The modal theories of a frame and pointed frame, as well as of classes of (pointed) Kripke structures or frames, are defined likewise.

A digression: Modal theories of models of set theory

Every model of (a sub-theory of) ZF can be regarded as a Kripke frame: $\mathfrak{M} = (U, \in)$. We call such frames $\in$-frames.

A generic problem: take a class of $\in$-frames and determine its modal theory, by means of a complete set of axioms.

For instance, what is the modal theory of Gödel's constructible universe L?

Modal equivalence

Two pointed Kripke τ-structures $(\mathfrak{M}, w)$ and $(\mathfrak{M}', w')$ are ML-equivalent, denoted

$$(\mathfrak{M}, w) \equiv_{\mathrm{ML}} (\mathfrak{M}', w'),$$

iff they satisfy exactly the same formulae of ML, i.e., iff

$$\mathrm{Th}_{\mathrm{ML}}(\mathfrak{M}, w) = \mathrm{Th}_{\mathrm{ML}}(\mathfrak{M}', w').$$

Modal equivalence between Kripke structures, frames, and pointed frames are defined likewise.

Modal equivalence and behavioral equivalence of Kripke structures

When are two Kripke structures modally equivalent?

van Benthem's sufficient condition (1976): when there is a zig-zag morphism between the Kripke structures.

Kripke structures can be viewed as transition systems.

In CS the question was raised: when are two transition systems 'behaviorally equivalent'?

Park 1981: characterizes behavioral equivalence by means of bisimulations.

Bisimulations and zig-zag morphisms are the same constructions.

Bisimulations between Kripke structures

Let $\mathfrak{M} = \langle W, \{R_\alpha\}_{\alpha\in\tau}, V\rangle$ and $\mathfrak{M}' = \langle W', \{R'_\alpha\}_{\alpha\in\tau}, V'\rangle$.

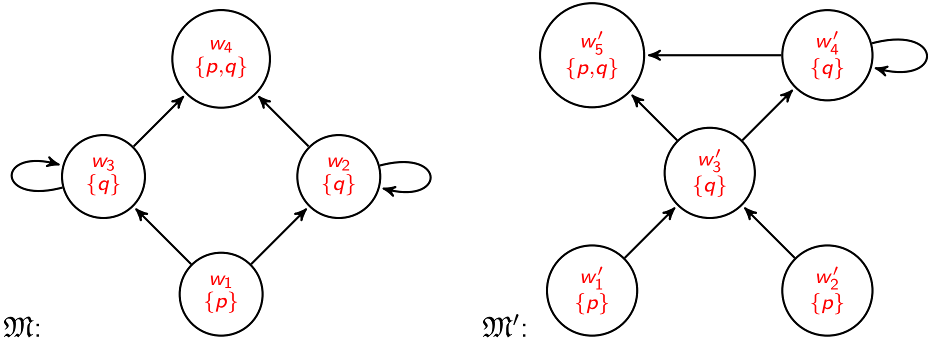
A bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$ is a non-empty relation $\rho \subseteq W \times W'$ satisfying the following conditions for any $w\rho w'$:

- Atom equivalence: w and w' satisfy the same atomic propositions, hereafter denoted by $w \simeq w'$.
- Forth: For any $\alpha \in \tau$, if $wR_\alpha u$ for some $u \in W$, then there is some $u' \in W'$ such that $w' R'_\alpha u'$ and $u\rho u'$.
- Back: Similarly, in the opposite direction: for any $\alpha \in \tau$, and $w' R'_\alpha u'$ there is some $u \in W$ such that $wR_\alpha u$ and $u\rho u'$.

If ρ is a bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$ we write $\rho\colon \mathfrak{M} \rightleftarrows \mathfrak{M}'$.

If, moreover, ρ is such that every element in $\mathfrak{M}$ is linked to some element of $\mathfrak{M}'$ and vice versa, we say that ρ is a global bisimulation and that $\mathfrak{M}$ and $\mathfrak{M}'$ are globally bisimilar.

Bisimulation: example



Claim: The relation ρ between $\mathfrak{M}$ and $\mathfrak{M}'$ consisting of all pairs of states from the respective structures, satisfying the same atomic propositions is a global bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$.

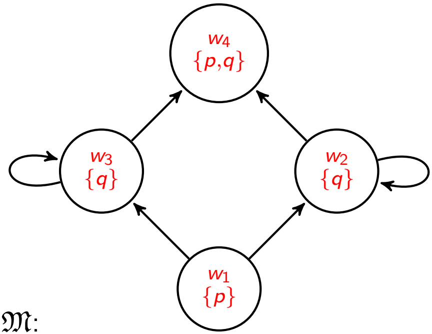
Bisimulations between pointed Kripke structures

Pointed Kripke structures $(\mathfrak{M}, w)$ and $(\mathfrak{M}', w')$ are locally bisimilar or locally bisimulation equivalent, denoted $(\mathfrak{M}, w) \rightleftarrows (\mathfrak{M}', w')$, if there is a bisimulation ρ between $\mathfrak{M}$ and $\mathfrak{M}'$ such that $w \rho w'$.

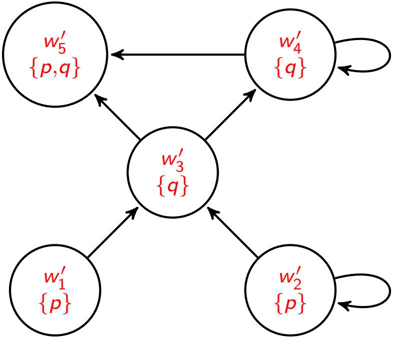
Bisimulations between (pointed) frames can be defined likewise, by omitting the atom equivalence condition.

Thus, a relation ρ is a bisimulation between two frames $\mathfrak{F}$ and $\mathfrak{F}'$, iff it is a bisimulation between the respective Kripke structures $\langle \mathfrak{F}, V_\perp \rangle$ and $\langle \mathfrak{F}', V'_\perp \rangle$ where the valuations $V_\perp$ and $V'_\perp$ render every atomic proposition false at every state of the respective frame.

Local bisimulation: example



$\mathfrak{M}$:



$\mathfrak{M}'$:

$\mathfrak{M}$ and $\mathfrak{M}'$ are not globally bisimular, because the state w_2' in $\mathfrak{M}'$ has no match in $\mathfrak{M}$ in any bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$.

However, the relation ρ between $\mathfrak{M}$ and $\mathfrak{M}'$ consisting of all pairs of states from the respective structures, excluding the state w_2', that satisfy the same atomic propositions is a local bisimulation between $(\mathfrak{M}, w_1)$ and $(\mathfrak{M}', w_1')$.