

Model Theory of Modal Logic  
Lecture 3

Valentin Goranko  
Technical University of Denmark

Third Indian School on Logic and its Applications  
Hyderabad, January 27, 2010

# Model Theory of Modal Logic

## Lecture 3

Valentin Goranko  
Technical University of Denmark

Third Indian School on Logic and its Applications  
Hyderabad, January 27, 2010

## Bisimulations between Kripke structures

Let  $\mathfrak{M} = \langle W, \{R_\alpha\}_{\alpha \in \tau}, V \rangle$  and  $\mathfrak{M}' = \langle W', \{R'_\alpha\}_{\alpha \in \tau}, V' \rangle$ .

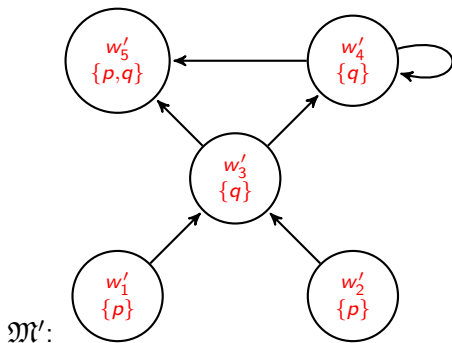
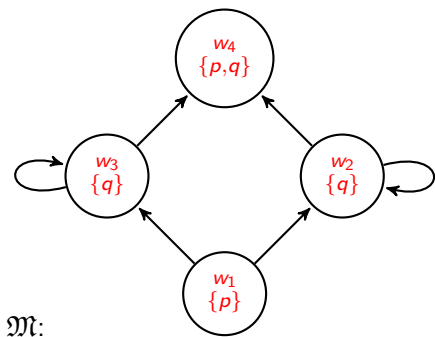
A **bisimulation** between  $\mathfrak{M}$  and  $\mathfrak{M}'$  is a non-empty relation  $\rho \subseteq W \times W'$  satisfying the following conditions for any  $w\rho w'$ :

- **Atomic equivalence:**  $w$  and  $w'$  satisfy the same atomic propositions, hereafter denoted by  $w \simeq w'$ .
- **Forth:** For any  $\alpha \in \tau$ , if  $wR_\alpha u$  for some  $u \in W$ , then there is some  $u' \in W'$  such that  $w'R'_\alpha u'$  and  $u\rho u'$ .
- **Back:** Similarly, in the opposite direction: for any  $\alpha \in \tau$ , and  $w'R'_\alpha u'$  there is some  $u \in W$  such that  $wR_\alpha u$  and  $u\rho u'$ .

If  $\rho$  is a bisimulation between  $\mathfrak{M}$  and  $\mathfrak{M}'$  we write  $\rho: \mathfrak{M} \rightleftharpoons \mathfrak{M}'$ .

If, moreover,  $\rho$  is such that every element in  $\mathfrak{M}$  is linked to some element of  $\mathfrak{M}'$  and vice versa, we say that  $\rho$  is a **global bisimulation** and that  $\mathfrak{M}$  and  $\mathfrak{M}'$  are **globally bisimilar**.

## Bisimulation: example



**Claim:** The relation  $\rho$  between  $\mathfrak{M}$  and  $\mathfrak{M}'$  consisting of all pairs of states from the respective structures, satisfying the same atomic propositions is a global bisimulation between  $\mathfrak{M}$  and  $\mathfrak{M}'$ .

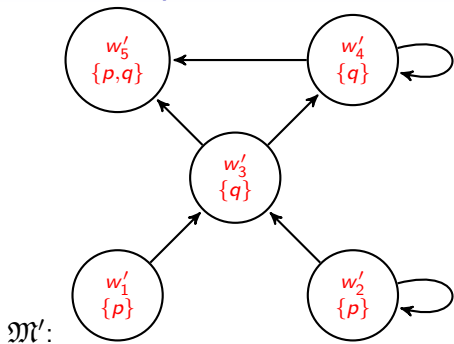
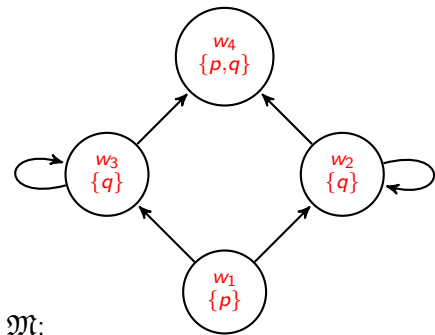
## Bisimulations between pointed Kripke structures

Pointed Kripke structures  $(\mathfrak{M}, w)$  and  $(\mathfrak{M}', w')$  are **locally bisimilar** or **locally bisimulation equivalent**, denoted  $(\mathfrak{M}, w) \rightleftharpoons (\mathfrak{M}', w')$ , if there is a bisimulation  $\rho$  between  $\mathfrak{M}$  and  $\mathfrak{M}'$  such that  $w\rho w'$ .

Bisimulations between (pointed) frames can be defined likewise, by omitting the atom equivalence condition.

Thus, a relation  $\rho$  is a bisimulation between two frames  $\mathfrak{F}$  and  $\mathfrak{F}'$ , iff it is a bisimulation between the respective Kripke structures  $\langle \mathfrak{F}, V_{\perp} \rangle$  and  $\langle \mathfrak{F}', V'_{\perp} \rangle$  where the valuations  $V_{\perp}$  and  $V'_{\perp}$  render every atomic proposition false at every state of the respective frame.

## Local bisimulation: example



$\mathfrak{M}$  and  $\mathfrak{M}'$  are not globally bisimilar, because the state  $w'_2$  in  $\mathfrak{M}'$  has no match in  $\mathfrak{M}$  in any bisimulation between  $\mathfrak{M}$  and  $\mathfrak{M}'$ .

However, the relation  $\rho$  between  $\mathfrak{M}$  and  $\mathfrak{M}'$  consisting of all pairs of states from the respective structures, excluding the state  $w'_2$ , that satisfy the same atomic propositions is a local bisimulation between  $(\mathfrak{M}, w_1)$  and  $(\mathfrak{M}', w'_1)$ .

## Bounded bisimulations

Let  $\mathfrak{M} = \langle W, \{R_\alpha\}_{\alpha \in \tau}, V \rangle$ ,  $\mathfrak{M}' = \langle W', \{R'_\alpha\}_{\alpha \in \tau}, V' \rangle$ ,  
and let  $w \in W, w' \in W'$ .

The property of a relation  $\rho \subseteq W \times W'$  to be a  **$k$ -bisimulation**  
between  $(\mathfrak{M}, w)$  and  $(\mathfrak{M}', w')$ , denoted  $(\mathfrak{M}, w) \stackrel{\rho}{\rightleftarrows}_k (\mathfrak{M}', w')$ , is  
defined inductively on  $k \in \mathbb{N}$  as follows:

- (**B<sub>0</sub>**)  $(\mathfrak{M}, w) \stackrel{\rho}{\rightleftarrows}_0 (\mathfrak{M}', w')$  iff  $w \rho w'$  and  $w \simeq w'$ .
- (**B<sub>k+1</sub>**)  $(\mathfrak{M}, w) \stackrel{\rho}{\rightleftarrows}_{k+1} (\mathfrak{M}', w')$  iff  $(\mathfrak{M}, w) \stackrel{\rho}{\rightleftarrows}_0 (\mathfrak{M}', w')$  and for any  
 $\alpha \in \tau$ :
- **Forth:** if  $wR_\alpha u$  for some  $u \in W$ , then there is some  $u' \in W'$   
such that  $w'R'_\alpha u'$  and  $(\mathfrak{M}, u) \stackrel{\rho}{\rightleftarrows}_k (\mathfrak{M}', u')$ .
  - **Back:** Similarly, if  $w'R'_\alpha u'$  for some  $u' \in W'$  then there is  
some  $u \in W$  such that  $wR_\alpha u$  and  $(\mathfrak{M}, u) \stackrel{\rho}{\rightleftarrows}_k (\mathfrak{M}', u')$ .

Clearly, every  $k$ -bisimulation between pointed Kripke structures is  
an  $m$ -bisimulation between them, for every  $m \leq k$ .

## Finite bisimilarity

Pointed Kripke structures  $(\mathfrak{M}, w)$  and  $(\mathfrak{M}', w')$  are  **$k$ -bisimilar**, or  **$k$ -bisimulation equivalent**, denoted  $(\mathfrak{M}, w) \Leftrightarrow_k (\mathfrak{M}', w')$ , if there is a  $k$ -bisimulation between them.

$(\mathfrak{M}, w)$  and  $(\mathfrak{M}', w')$  are **finitely bisimilar**, or **finite bisimulation equivalent**, denoted  $(\mathfrak{M}, w) \Leftrightarrow_f (\mathfrak{M}', w')$ , if they are  $k$ -bisimilar for every  $k \geq 0$ .

NB: the  $k$ -bisimulations required to exist between finitely bisimilar pointed Kripke structures need not agree with each other.

$k$ -bisimilar and finitely bisimilar pointed Kripke frames are defined likewise.

Clearly, every two bisimilar pointed Kripke structures are finitely bisimilar.

**Question:** Is the converse always true?

## Modal nesting depth of a formula

The **nesting depth**  $\delta$  of a modal formula is defined recursively as follows:

$$\delta(\perp) = \delta(p) = 0;$$

$$\delta(\phi_1 \rightarrow \phi_2) = \max(\delta(\phi_1), \delta(\phi_2));$$

$$\delta(\langle \alpha \rangle \phi) = \delta(\phi) + 1.$$

The fragment **ML<sub>n</sub>( $\tau$ )** comprises all formulae of **ML( $\tau$ )** with nesting depth  $\leq n$ .

## Invariance of modal formulae under bisimulations

We hereby omit the type  $\tau$  from the notation.

### Theorem (bounded bisimulation invariance)

For every  $n \in \mathbb{N}$ :

if  $(\mathfrak{M}, w) \xleftrightarrow{n} (\mathfrak{M}', w')$  then  $(\mathfrak{M}, w) \equiv_{\text{ML}_n} (\mathfrak{M}', w')$ .

Proof: Induction on  $n$ . Exercise.

### Corollary (bisimulation invariance)

ML is bisimulation invariant:

if  $(\mathfrak{M}, w) \xleftrightarrow{\infty} (\mathfrak{M}', w')$  then  $(\mathfrak{M}, w) \equiv_{\text{ML}} (\mathfrak{M}', w')$ .

### Corollary

For every constant formula  $\theta \in \text{ML}$  and pointed Kripke frames  $(\mathfrak{F}, w)$  and  $(\mathfrak{F}', w')$ :

if  $(\mathfrak{F}, w) \xleftrightarrow{\infty} (\mathfrak{F}', w')$ , then  $(\mathfrak{F}, w) \models \theta$  iff  $(\mathfrak{F}', w') \models \theta$ .

## Special case of bisimulations: bounded morphisms

### Definition

Let  $\mathfrak{M} = \langle W, \{R_\alpha\}_{\alpha \in \tau}, V \rangle$  and  $\mathfrak{M}' = \langle W', \{R'_\alpha\}_{\alpha \in \tau}, V' \rangle$  be Kripke structures. A function  $\rho: W \rightarrow W'$  is a **bounded morphism** from  $\mathfrak{M}$  to  $\mathfrak{M}'$ , denoted  $\rho: \mathfrak{M} \xrightarrow{\text{b}} \mathfrak{M}'$ , if its graph is a bisimulation between  $\mathfrak{M}$  and  $\mathfrak{M}'$ .

If  $\rho$  is onto, then  $\mathfrak{M}'$  is a **bounded morphic image** of  $\mathfrak{M}$ .

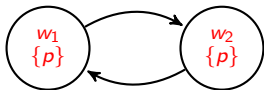
Thus, for each  $u \in W$ , a bounded morphism  $\rho$  uniquely singles out a bisimilar state  $\rho(w)$  in  $W'$ . The bisimulation conditions for a bounded morphism become respectively:

**Atomic equivalence:**  $w \simeq \rho(w)$  for every  $w \in W$ .

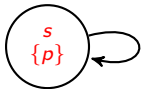
**Forth:** For any  $w \in W$  and  $\alpha \in \tau$ , if  $wR_\alpha u$  for some  $u \in W$ , then  $\rho(w)R'_\alpha \rho(u)$ .

**Back:** For any  $w \in W$  and  $\alpha \in \tau$ , if  $\rho(w)R'_\alpha u'$  for some  $u' \in W'$  then  $u' = \rho(u)$  for some  $u \in W$  such that  $wR_\alpha u$ .

## Bounded morphisms: example 1

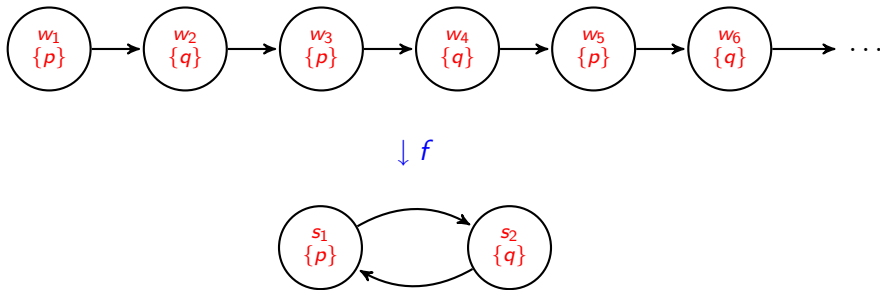


$\downarrow f$



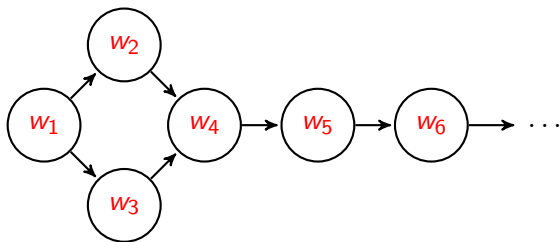
$$f(w_1) = f(w_2) = s.$$

## Bounded morphisms: example 2

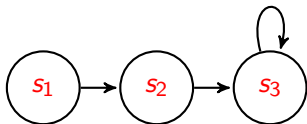


$$f(w_{2k-1}) = s_1, f(w_{2k}) = s_2, \text{ for } k > 0.$$

## Bounded morphisms: example 3



$\downarrow f$



$f(w_1) = s_1$ ,  $f(w_2) = f(w_3) = s_2$ ,  $f(w_k) = s_3$  for  $k \geq 4$ .

# Bounded morphisms preserve truth of modal formulae

## Corollary

If  $\rho: \mathfrak{M} \xrightarrow{\cong} \mathfrak{M}'$  is a bounded morphism and  $\phi \in ML$ , then:

1. for all  $u \in \text{dom}(\mathfrak{F})$ :  $\mathfrak{M}, u \models \phi$  iff  $\mathfrak{M}', \rho(u) \models \phi$ .
2. If  $\rho$  is onto, then  $\mathfrak{M} \models \phi$  iff  $\mathfrak{M}' \models \phi$ , i.e.,  
 $\text{Th}_{ML}(\mathfrak{M}) = \text{Th}_{ML}(\mathfrak{M}')$ .
3. If  $\mathfrak{F}, u \models \phi$  then  $\mathfrak{F}', \rho(u) \models \phi$ .
4. If  $\rho$  is onto, then  $\mathfrak{F} \models \phi$  implies  $\mathfrak{F}' \models \phi$ .

## Generated substructures

If  $R \subseteq W^2$  and  $W' \subseteq W$  denote  $R \upharpoonright W' = R \cap (W' \times W')$ .

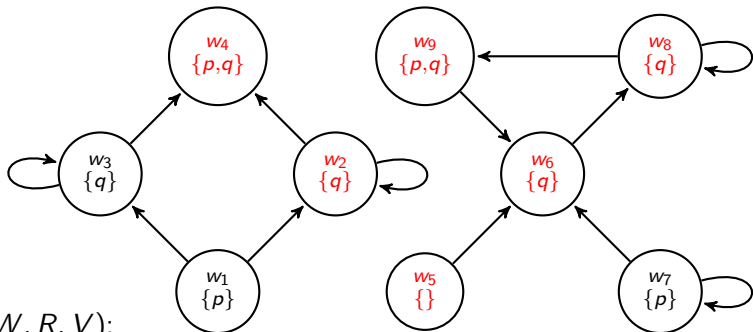
If  $f$  is a mapping defined on  $W$ ,  $f \upharpoonright W'$  is the restriction of  $f$  to  $W'$ .

### Definition

Let  $\mathfrak{F} = \langle W, \{R_\alpha\}_{\alpha \in \mathcal{T}} \rangle$  be a Kripke frame, respectively  $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$  a Kripke structure over  $\mathfrak{F}$ , and  $W' \subseteq W$ .

- (i) The induced **subframe of  $\mathfrak{F}$  over  $W'$**  is the Kripke frame  $\mathfrak{F}' := \mathfrak{F} \upharpoonright W' = \langle W', \{R_\alpha \upharpoonright W'\}_{\alpha \in \mathcal{T}} \rangle$ .  
The subframe relationship is denoted  $\mathfrak{F}' \leq \mathfrak{F}$ .
- (ii)  $\mathfrak{F}' = \mathfrak{F} \upharpoonright W'$  is a **generated subframe of  $\mathfrak{F}$** , denoted  $\mathfrak{F}' \trianglelefteq \mathfrak{F}$ , if  $W'$  is closed under all accessibility relations in the sense that  $wR_\alpha u$  for  $w \in W'$  implies  $u \in W'$ .
- (iii) The induced **substructure of  $\mathfrak{M}$  over  $W'$**  is the Kripke structure  $\mathfrak{M}' = \mathfrak{M} \upharpoonright W' = \langle \mathfrak{F} \upharpoonright W', V \upharpoonright W' \rangle$ , denoted  $\mathfrak{M}' \leq \mathfrak{M}$ .  
If  $\mathfrak{F} \upharpoonright W' \trianglelefteq \mathfrak{F}$ , then  $\mathfrak{M}'$  is a **generated substructure of  $\mathfrak{M}$** , denoted  $\mathfrak{M}' \triangleleft \mathfrak{M}$ .

## Generated substructure: example



$\mathfrak{M} = (W, R, V)$ :

Let  $W' = \{w_2, w_4, w_5, w_6, w_8, w_9\}$  and  $\mathfrak{M}' = \mathfrak{M} \upharpoonright W'$  is a generated substructure.

# Generated substructures preserve truth of modal formulae

## Proposition

For any Kripke structures  $\mathfrak{M}' \trianglelefteq \mathfrak{M}$  and formula  $\phi$  of ML:

- (i) for every  $u \in \text{dom}(\mathfrak{M}')$ :  $\mathfrak{M}, u \models \phi$  iff  $\mathfrak{M}', u \models \phi$ .
- (ii)  $\mathfrak{M} \models \phi$  implies  $\mathfrak{M}' \models \phi$ .

Likewise, for frames  $\mathfrak{F}' \trianglelefteq \mathfrak{F}$  and  $u \in \text{dom}(\mathfrak{F}')$ :

$\mathfrak{F}, u \models \phi$  iff  $\mathfrak{F}', u \models \phi$ , and  $\mathfrak{F} \models \phi$  implies  $\mathfrak{F}' \models \phi$ .

## Paths in Kripke structures

A **path** in a Kripke frame  $\mathfrak{F} = \langle W, \{R_\alpha\}_{\alpha \in \tau} \rangle$  is a sequence  $\vec{w} = (w_0, \alpha_1, w_1, \dots, \alpha_k, w_k)$ , where  $w_{i-1} R_{\alpha_i} w_i$  for  $i = 1, \dots, k$ . This path is **rooted** at  $w_0$  and has **length**  $k$ .

A path of length  $k = 0$ ,  $\vec{w} = (w_0)$ , is identified with its root  $w_0$ .

For  $u \in W$ , we denote the set of all paths rooted at  $u$  by  $\vec{W}[u]$ .

For every path  $\vec{w} = (w_0, \alpha_1, w_1, \dots, \alpha_k, w_k)$  we define the 'terminal state' function  $f(\vec{w}) = w_k$ .

Then we define

$$W[u] := \{f(\vec{w}) \mid \vec{w} \in \vec{W}[u]\}$$

Thus,  $W[u]$  is the set of all states in  $\mathfrak{F}$  reachable from  $u$  (including  $u$  itself).

## Rooted substructures

### Definition

Let  $\mathfrak{F} = \langle W, \{R_\alpha\}_{\alpha \in \tau} \rangle$  be a Kripke frame,  $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$  a Kripke structure over  $\mathfrak{F}$ , and  $u \in W$ .

- (i) The **subframe of  $\mathfrak{F}$  rooted at  $u$**  is the frame

$$\mathfrak{F}[u] = \mathfrak{F} \upharpoonright W[u].$$

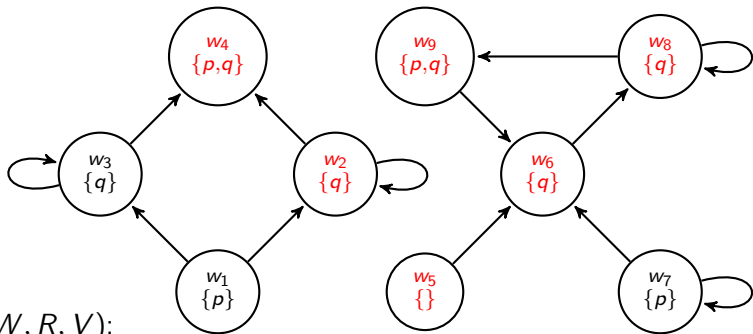
- (ii) The **substructure of  $\mathfrak{M}$  rooted at  $u$**  is the Kripke structure

$$\mathfrak{M}[u] = \mathfrak{M} \upharpoonright W[u].$$

- (iii)  $\mathfrak{F}$  (respectively  $\mathfrak{M}$ ) is **rooted at  $u$**  if  $W[u] = W$ .

Clearly, for any  $u \in W$ :  $\mathfrak{F}[u] \trianglelefteq \mathfrak{F}$  and  $\mathfrak{M}[u] \trianglelefteq \mathfrak{M}$ .

## Rooted substructure: examples



$\mathfrak{M} = (W, R, V)$ :

$\mathfrak{M}[w_2]$  contains  $\{w_2, w_4\}$ .

$\mathfrak{M}[w_5]$  contains  $\{w_5, w_6, w_8, w_9\}$ .

## Rooted substructures preserve truth of modal formulae

Rooted substructures are generated substructures.

### Corollary

*For every Kripke structure  $\mathfrak{M}' \trianglelefteq \mathfrak{M}$  and formula  $\phi$  of  $ML(\tau)$ :*

- (i) *for all  $u \in W$ :  $\mathfrak{M}, u \models \phi$  iff  $\mathfrak{M}[u], u \models \phi$ .*
- (ii)  *$\mathfrak{M} \models \phi$  implies  $\mathfrak{M}[u] \models \phi$ .*
- (iii) *Likewise for (pointed) frames.*

Thus, any satisfiable formula is satisfiable at the root of a rooted Kripke structure.

## A refinement: depth $n$ rooted substructures

Let  $\mathfrak{F} = \langle W, \{R_\alpha\}_{\alpha \in \tau} \rangle$  be a Kripke frame and  $u \in W$ . For any  $n \in \mathbb{N}$  denote by  $W^n[u]$  the set of paths of length at most  $n$  in  $\mathfrak{F}$ .

For any Kripke structure  $\mathfrak{M} = (\mathfrak{F}, V)$ , the **depth  $n$  substructure of  $\mathfrak{M}$  rooted at  $u$**  is the Kripke structure

$$\mathfrak{M}^n[u] = \mathfrak{M} \upharpoonright W^n[u].$$

Note that the embedding of  $\mathfrak{M}^n[u]$  into  $\mathfrak{M}[u]$  is an  $n$ -bisimulation between  $\mathfrak{M}^n[u], u$  and  $\mathfrak{M}[u], u$ .

### Corollary

*For every Kripke structure  $\mathfrak{M}$ , a state  $u \in \mathfrak{M}$ , and a formula  $\phi$  of nesting depth  $n$ :*

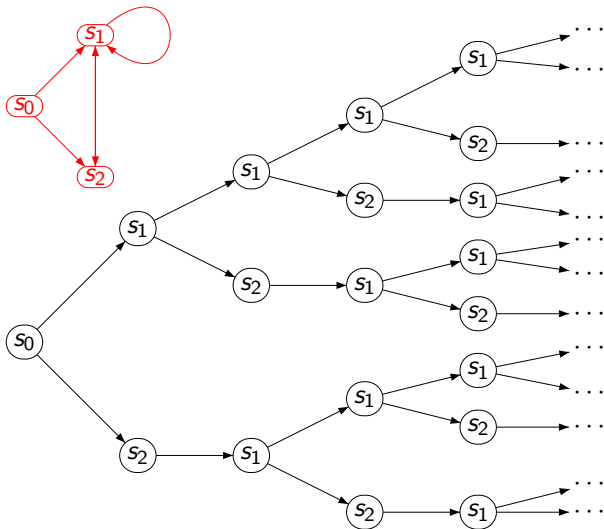
$$\mathfrak{M}^n[u], u \models \phi \text{ iff } \mathfrak{M}[u], u \models \phi \text{ iff } \mathfrak{M}, u \models \phi.$$

Thus, to check the truth of a formula of nesting depth  $n$  at a given state  $u$  of a Kripke structure  $\mathfrak{M}$  it suffices to consider the depth  $n$  substructure of  $\mathfrak{M}$  rooted at  $u$ .

## Unfolding of a Kripke structure, intuitively

- Unfolding of a pointed Kripke structure  $(\mathfrak{K}, s)$  starts from the root  $s$ , produces new copies of all successors of that state, and inserts respective transitions to these copies.
- Then the procedure continues recursively from each of these successor states.
- The result is a (usually infinite) tree in which all paths starting from the root of the original Kripke structure are presented explicitly as branches.

# Unfolding of a pointed Kripke structure: example



## Unfolding of a Kripke structure, formally

Recall the function  $f: \vec{W}[u] \rightarrow W$ , which maps the path  $\vec{w} = (u = w_0, \alpha_1, w_1, \dots, \alpha_k, w_k)$  to its terminal state  $f(\vec{w}) = w_k$ .

The unfolding of  $\vec{\mathfrak{M}}[u]$  of  $\mathfrak{M}$  at  $u$  is based on the set  $\vec{W}[u]$  of all paths rooted at  $u$ , with the natural definition of accessibility relations and a valuation that turns  $f$  into a bounded morphism.

### Definition

The **unfolding** of  $\mathfrak{M} = \langle W, \{R_\alpha\}_{\alpha \in \tau}, V \rangle$  from a state  $u \in W$  is the rooted Kripke structure  $\vec{\mathfrak{M}}[u] := \langle \vec{W}[u], \{\vec{R}_\alpha\}_{\alpha \in \tau}, \vec{V} \rangle$  with root  $u = (u)$ , where

$$\vec{R}_\alpha := \{(\vec{w}, (\vec{w}, \alpha, w')) \mid \vec{w} \in \vec{W}[u], f(\vec{w})R_\alpha w'\},$$

$$\vec{V}(p) := f^{-1}[V(p)].$$

## Unfoldings are trees

The unfolding  $\vec{\mathfrak{M}}[u]$  is a tree-like structure with root  $u = (u)$  in the sense of the following definition.

### Definition

A pointed frame  $\mathfrak{F} = \langle W, \{R_\alpha\}_{\alpha \in \tau} \rangle$  with distinguished state  $u \in W$  is a **tree** with root  $u$  if  $\mathfrak{F}$  is rooted at  $u$  and every state  $w \in W$  is reachable from  $u$  by a unique path.

Accordingly, every Kripke structure over  $(\mathfrak{F}, u)$  is a tree structure.

### Fact

*For any pointed Kripke structure  $(\mathfrak{M}, u)$  the map  $f: \vec{W}[u] \rightarrow W[u]$  is a bounded morphism of the unfolding  $\vec{\mathfrak{M}}[u]$  onto  $\mathfrak{M}[u]$ .*

### Corollary (tree-model property)

*Every satisfiable modal formula is satisfiable at the root of a tree.*

**Challenge** Prove that every satisfiable modal formula is satisfiable at the root of a **finite tree**.

## Disjoint unions

The **disjoint union** of an arbitrary family  $\{W^i\}_{i \in I}$  of (not necessarily disjoint) sets is

$$\bigsqcup_{i \in I} W^i := \bigcup_{i \in I} (W^i \times \{i\}).$$

### Definition

Consider a family of  $\tau$ -frames  $\{\mathfrak{F}^i = \langle W^i, \{R_\alpha\}_{\alpha \in \tau} \rangle\}_{i \in I}$  and a family of Kripke structures  $\{\mathfrak{M}^i = \langle \mathfrak{F}^i, V^i \rangle\}_{i \in I}$  over these.

- (i) The **disjoint union of  $\{\mathfrak{F}^i\}_{i \in I}$**  is the frame  $\bigsqcup_{i \in I} \mathfrak{F}^i = \langle \bigsqcup_{i \in I} W^i, \{R_\alpha\}_{\alpha \in \tau} \rangle$ , where  $(w_0, i_0)R_\alpha(w_1, i_1)$  iff  $i_0 = i_1 = i$  and  $w_0 R_\alpha^i w_1$ .
- (ii) The **disjoint union of  $\{\mathfrak{M}^i\}_{i \in I}$**  is the Kripke  $\tau$ -structure  $\bigsqcup_{i \in I} \mathfrak{M}^i = \langle \bigsqcup_{i \in I} \mathfrak{F}^i, V \rangle$  where  $V(p) = \bigsqcup_{i \in I} V^i(p)$ .

## Component embeddings into disjoint unions

There is a natural injection (embedding)

$$\epsilon_j: W^j \rightarrow \bigcup_{i \in I} (W^i \times \{i\})$$

of each component set into the disjoint union, which maps  $w \in W^j$  to  $(w, j) \in \bigcup_{i \in I} (W^i \times \{i\})$ .

That natural injection  $\epsilon_j$  isomorphically embeds  $\mathfrak{M}^j$  into  $\biguplus_{i \in I} \mathfrak{M}^i$ .  
Moreover, it is a bounded morphism with image  $\epsilon_j[\mathfrak{M}^j] \simeq \mathfrak{M}^j$  and  $\epsilon_j[\mathfrak{M}^j] \trianglelefteq \biguplus_{i \in I} \mathfrak{M}^i$ .

## Disjoint unions preserve the truth of modal formulae

### Proposition

Given a family of  $\tau$ -frames  $\{\mathfrak{F}^i = \langle W^i, \{R_\alpha^i\}_{\alpha \in \tau} \rangle\}_{i \in I}$ , a family of Kripke structures  $\{\mathfrak{M}^i = \langle \mathfrak{F}^i, V^i \rangle\}_{i \in I}$  over these frames, and  $\phi \in \text{ML}(\tau)$ :

1. For every  $j \in I$  and  $w \in \text{dom}(\mathfrak{M}^j)$ :  
 $\mathfrak{M}^j, w \models \phi$  iff  $\biguplus_{i \in I} \mathfrak{M}^i, (w, j) \models \phi$ .
2. For every  $j \in I$  and  $w \in \text{dom}(\mathfrak{F}^j)$ :  
 $\mathfrak{F}^j, w \models \phi$  iff  $\biguplus_{i \in I} \mathfrak{F}^i, (w, j) \models \phi$ .
3.  $\biguplus_{i \in I} \mathfrak{M}^i \models \phi$  iff  $\mathfrak{M}^i \models \phi$  for every  $i \in I$ .
4.  $\biguplus_{i \in I} \mathfrak{F}^i \models \phi$  iff  $\mathfrak{F}^i \models \phi$  for every  $i \in I$ .

## A structural characterization of Kripke structures

### Proposition (van Benthem, 1983)

*Any Kripke structure is a bounded morphic image of a disjoint union of rooted Kripke structures, and indeed of tree structures.*

#### PROOF SKETCH

Consider the family  $\{\mathfrak{M}[u]\}_{u \in W}$  (respectively,  $\{\vec{\mathfrak{M}}[u]\}_{u \in W}$ ).

Take its disjoint union  $\mathfrak{U}$ .

For every component of  $\mathfrak{U}$  take the inverse mapping of the natural embedding of the respective rooted substructure  $\mathfrak{M}[u]$  into  $\mathfrak{U}$ .

The union of these mappings is the desired bounded morphism from  $\mathfrak{U}$  onto  $\mathfrak{M}$ .