Model Theory of Modal Logic
Lecture 4

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Proving modal non-definability: the generic model-theoretic approach

To show that a given property of structures is definable in a given logic, it suffices simply to find a defining formula in that logic.

Showing that a property is not definable, however, usually requires elaborate model-theoretic arguments.

A standard method for establishing non-definability of a property $\mathcal{P}$ (i.e., of the class of structures satisfying that property) in a logic $\mathcal{L}$ is to show that $\mathcal{P}$ is not closed under some construction that preserves truth (validity) of all formulae of $\mathcal{L}$.

This works particularly well in the case of modal logic.

Modal non-definability in pointed Kripke structures

On pointed Kripke structures, modal formulae capture only local properties: whether or not $\mathfrak{M}, w \models \varphi$ only depends on $(\mathfrak{M}[w], w)$.

In other words, modal formulae are incapable of expressing any property of $(\mathfrak{M}, w)$ that involves points beyond $\mathfrak{M}[w]$.

Example There is no $\varphi \in \mathrm{ML}$ such that $\mathfrak{M}, w \models \varphi$ iff $\mathfrak{M} \models p$.

Indeed, one can add to $\mathfrak{M}$ an extra point, not reachable from w, where p is false. The resulting pointed structure $(\mathfrak{M}', w)$ is locally bisimilar to $(\mathfrak{M}, w)$, whence φ would have to be equally true or false at w in both.

Modal non-definability in Kripke structures

Example. There is no $\varphi \in \mathrm{ML}$ such that $\mathfrak{M} \models \varphi$ iff the accessibility relation in $\mathfrak{M}$ is reflexive.

This follows for instance from the fact that the unfolding of any frame is irreflexive. If $\mathfrak{M}$ is reflexive, then so is the generated substructure $\mathfrak{M}[u]$ which, however, is also a bounded morphic image of the irreflexive $\vec{\mathfrak{M}}[u]$.

Reflexivity, however, is well-known to be definable in terms of frame validity by the modal formula $\Box p \rightarrow p$; equivalently, by the second-order sentence $\forall P \forall x (\forall y (Rxy \rightarrow Py) \rightarrow Px)$.

Intuitively, in terms of truth in Kripke structures, modal formulae can make very little reference to the underlying frame.

Modal non-definability in Kripke frames
Simple examples

Example. Irreflexivity is not a modally definable frame property.

Proof: Irreflexivity is not preserved under surjective bounded morphisms, while they preserve frame validity.

For instance, consider the irreflexive frame $\mathfrak{F} = \langle \{w_1, w_2\}, \{(w_1, w_2), (w_2, w_1)\} \rangle$ and its bounded morphic image $\mathfrak{F}' = \langle \{w\}, \{(w, w)\} \rangle$, which is reflexive. Any modal formula ϕ valid in the former would also be valid in the latter.

Example. The class of non-reflexive frames (having at least one irreflexive point) is not modally definable, either, because it is not closed under passage to generated subframes.

Example. The classes of finite frames, connected frames, or of frames with a universal accessibility relation ($R = W^2$), are not modally definable, as they are not closed under disjoint unions.

Modal non-definability in Kripke frames
More examples

Example. The property of a frame to be a reflexive partial ordering is not modally definable.

Proof: Anti-symmetry is not preserved under bounded morphisms.

Indeed, $\langle \mathbb{Z}, \leq \rangle$ is antisymmetric, but it can be mapped by a surjective bounded morphism onto the symmetric frame $\mathfrak{F}$, sending all odd numbers to w_1 and all even ones to w_2.

Example. The property of continuity, aka Dedekind completeness is not modally definable in modal logic.

Proof: Follows from the (non-trivial) fact that $\langle \mathbb{R}, \leq \rangle$ (which is continuous) and $\langle \mathbb{Q}, \leq \rangle$ (which is not) have the same modal theory.

On the other hand, continuity is definable in the basic temporal logic by the formula $\Box([P]p \to \langle F \rangle [P]p) \to ([P]p \to [F]p)$, where F and P are respectively the future and past modality, and $\Box\varphi = [P]\varphi \wedge \varphi \wedge [F]\varphi$ is the always modality.

Modal non-definability in Kripke frames
More preservation results are needed

Preservation under generated subframes, bounded morphic images and disjoint unions is not sufficient to guarantee modal definability in terms of frame validity, even for first-order definable properties.

For instance, the class of frames defined by the first-order sentence $\forall x \exists y (xRy \wedge yRy)$ is not modally definable, despite being closed under these three constructions.

However, this formula is not reflected by ultrafilter extensions of Kripke frames, which reflect the validity of every modal formula.

Bisimulation games: the setup

$\mathfrak{M}_1 = (W_1, R_1, V_1)$ and $\mathfrak{M}_2 = (W_2, R_2, V_2)$: Kripke structures (KS) of the same type.

Bisimulation game on $\mathfrak{M}_1$ and $\mathfrak{M}_2$:

- played by two players **I** (the Challenger) and **II** (the Defender).
- with two pebbles, one in $\mathfrak{M}_1$ and one in $\mathfrak{M}_2$, to mark the 'current state' in each structure.

Configuration in the game: pair of pointed Kripke structures

$$(\mathfrak{M}_1, s_1; \mathfrak{M}_2, s_2).$$

The distinguished points are the current positions of the two pebbles.

Bisimulation games: playing the game

The game starts from initial configuration and is played in rounds.

In each round player **I** selects a pebble and moves it forward along a transition in the respective structure, to a successor state.

Then player **II** responds by similarly moving forward the pebble in the other structure along a transition with the same label.

The objective of player **I**: to exhibit a behavioral difference between the two pointed Kripke structures in the initial configuration by choosing a sequence of transition in one of them that cannot be properly simulated in the other.

The objective of player **II**: to defend the claim that the two pointed Kripke structures in the initial configuration are bisimilar, by replying with transitions maintaining the bisimulation throughout the game.

Bisimulation games: the winning condition

During the game player **II** loses if she cannot respond correctly to the move of player **I**, or if the two pebble positions in the resulting new configuration do not match on some atomic proposition.

On the other hand, player **I** loses during the game if he cannot make a move in the current round because both pebbles are in states without successors.

The bisimulation game can be played for a pre-determined number of rounds, or indefinitely.

The n-round bisimulation game terminates after n rounds, or earlier if either player loses during one of these rounds.
If the n-th round is completed without violating the atom equivalence in any configuration, player **II** wins the game.

Respectively, if player **II** can play the infinite bisimulation game forever without loosing at any round, she wins the game.

Bisimulation games: winning strategies

Player **II** has a winning strategy in a given bisimulation game if she has responses to any challenges of Player **I** that guarantee her to win the game. Winning strategy of player **I** is defined likewise.

Proposition Every bisimulation game is determined, i.e., one of the players has a winning strategy.

Winning bisimulation games and bisimulation equivalence

Theorem

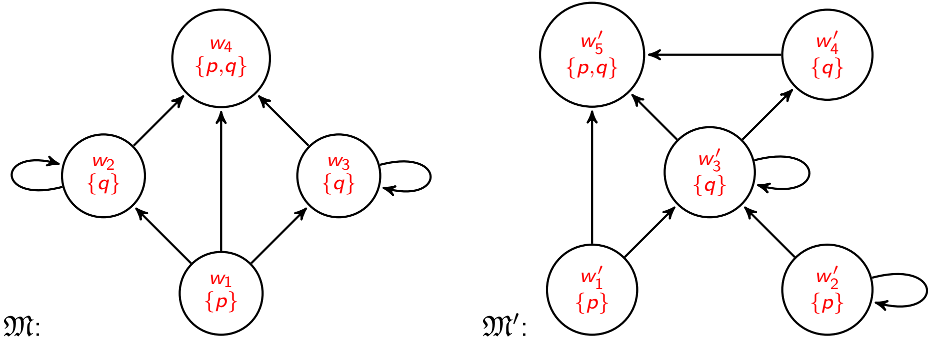
1. Player **II** has a winning strategy in the *n*-round bisimulation game with initial configuration $(\mathfrak{M}_1, s_1; \mathfrak{M}_2, s_2)$ if and only if $(\mathfrak{M}_1, s_1) \rightleftarrows_n (\mathfrak{M}_2, s_2)$.
2. Player **II** has a winning strategy in the unbounded bisimulation game with initial configuration $(\mathfrak{M}_1, s_1; \mathfrak{M}_2, s_2)$ if and only if $(\mathfrak{M}_1, s_1) \rightleftarrows (\mathfrak{M}_2, s_2)$.

Proof sketch:

On the one hand, any bisimulation $\rho\colon (\mathfrak{M}, w) \rightleftarrows (\mathfrak{M}', w')$ is a non-deterministic winning strategy for **II**: she merely needs to select her responses so that the pebbled states remain linked by ρ. The atom equivalence condition on ρ guarantees that atom equivalence between pebbled states is maintained; the forth and back conditions guarantee matching responses to challenges played by **I** respectively in $\mathfrak{M}$ and in $\mathfrak{M}'$.

Conversely, the set of pairs (u, u') in all configurations from which **II** has a winning strategy is a bisimulation.

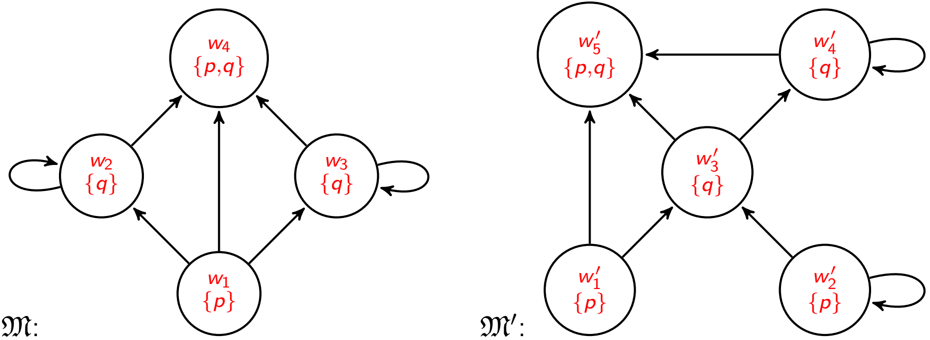
Bounded bisimulation games: example



Who has a winning strategy in the 2-round bisimulation game with initial configuration $(\mathfrak{M}, w_1; \mathfrak{M}', w'_1)$?

Who has a winning strategy in the 3-round bisimulation game with initial configuration $(\mathfrak{M}, w_1; \mathfrak{M}', w'_1)$?

Unbounded bisimulation games: example



Who has a winning strategy in the unbounded bisimulation game with initial configuration $(\mathfrak{M}, w_1; \mathfrak{M}', w'_1)$?

ML-equivalence of pointed Kripke structures

Recall:

Definition (ML-equivalence)

Two pointed Kripke structures $(\mathfrak{M}_1, w_1)$ and $(\mathfrak{M}_2, w_2)$ are ML-equivalent, denoted

$$(\mathfrak{M}_1, w_1) \equiv_{\mathrm{ML}} (\mathfrak{M}_2, w_2),$$

if they satisfy the same ML-formulae.

$(\mathfrak{M}_1, w_1)$ and $(\mathfrak{M}_2, w_2)$ are ML^n-equivalent, denoted

$$(\mathfrak{M}_1, w_1) \equiv_{\mathrm{ML}^n} (\mathfrak{M}_2, w_2),$$

if they satisfy the same formulae of ML^n.

Invariance of ML-formulae under bisimulations

Recall:

Theorem

The formulae of ML^n are invariant under n-bisimulations:

If $(\mathfrak{M}_1, w_1)$ and $(\mathfrak{M}_2, w_2)$ are pointed Kripke structures and $n \in \mathbb{N}$ is such that $(\mathfrak{M}_1, w_1) \rightleftarrows_n (\mathfrak{M}_2, w_2)$, then $(\mathfrak{M}_1, w_1) \equiv_{ML^n} (\mathfrak{M}_2, w_2)$.

Corollary

The formulae of ML are invariant under bisimulations:

If $(\mathfrak{M}_1, w_1)$ and $(\mathfrak{M}_2, w_2)$ are pointed Kripke structures, such that $(\mathfrak{M}_1, w_1) \rightleftarrows (\mathfrak{M}_2, w_2)$, then $(\mathfrak{M}_1, w_1) \equiv_{ML} (\mathfrak{M}_2, w_2)$.

Characteristic formulae

Hereafter we assume AP to be finite.

With every pointed Kripke structures $(\mathfrak{M}, w)$, where $\mathfrak{M} = (W, R, V)$ and $n \in \mathbb{N}$ we associate a characteristic formula of depth n, $\chi^n_{[\mathfrak{M},w]} \in \mathrm{ML}^n$, defined inductively as follows:

- $\chi^0_{[\mathfrak{M},w]} := \bigwedge\{p \mid w \in V(p)\} \wedge \bigwedge\{\neg p \mid w \notin V(p)\}$, where p ranges over AP.
- $\chi^{n+1}_{[\mathfrak{M},w]} := \chi^0_{[\mathfrak{M},w]} \wedge \bigwedge_{wRt} \Diamond\chi^n_{[\mathfrak{M},t]} \wedge \Box \bigvee_{wRt} \chi^n_{[\mathfrak{M},t]}$.

Note that there are only finitely many different (up to logical equivalence) characteristic formulae of depth n, so even though a state w may have infinitely many successors, every formula $\chi^n_{[\mathfrak{M},w]}$ is well-defined, i.e., finite.

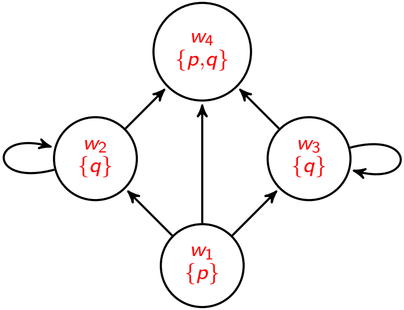
Proposition

$\mathfrak{M}, w \models \chi^n_{[\mathfrak{M},w]}$, *for every* $n \in \mathbb{N}$.

Proof: Induction on n, simultaneously for all $w \in \mathfrak{M}$. Exercise.

Characteristic formulae: example

$\mathfrak{M}$:



Let $\mathrm{AP} = \{p, q\}$. Then:

$$\chi^0_{[\mathfrak{M},w_1]} = p \wedge \neg q, \quad \chi^0_{[\mathfrak{M},w_2]} = \chi^0_{[\mathfrak{M},w_3]} = \neg p \wedge q; \; \chi^0_{[\mathfrak{M},w_4]} = p \wedge q.$$

$$\chi^1_{[\mathfrak{M},w_1]} = (p \wedge \neg q) \wedge \Diamond(\neg p \wedge q) \wedge \Diamond(p \wedge q) \wedge \Box((\neg p \wedge q) \vee (p \wedge q)).$$

$$\chi^1_{[\mathfrak{M},w_2]} = (\neg p \wedge q) \wedge \Diamond(\neg p \wedge q) \wedge \Diamond(p \wedge q) \wedge \Box((\neg p \wedge q) \vee (p \wedge q)).$$

$$\chi^1_{[\mathfrak{M},w_2]} = \chi^1_{[\mathfrak{M},w_3]}; \quad \chi^1_{[\mathfrak{M},w_4]} = (p \wedge q) \wedge \Box\bot.$$

$$\chi^2_{[\mathfrak{M},w_1]} = (p \wedge \neg q) \wedge \Diamond\chi^1_{[\mathfrak{M},w_2]} \wedge \Diamond\chi^1_{[\mathfrak{M},w_4]} \wedge \Box(\chi^1_{[\mathfrak{M},w_2]} \vee \chi^1_{[\mathfrak{M},w_4]})$$

Characteristic formulae and bisimulation games

Theorem

For every pointed Kripke structures $(\mathfrak{M}, w)$ and $(\mathfrak{M}', w')$ the following are equivalent:

1. $\mathfrak{M}', w' \models \chi^n_{[\mathfrak{M}, w]}$.
2. Player **II** has a winning strategy in the n-round game from $(\mathfrak{M}, w; \mathfrak{M}', w')$.

Proof sketch: The implication $2 \Rightarrow 1$ is immediate, because the winning strategy of player **II** implies n-bisimilarity of the pointed structures in the starting configuration. For the converse, we show by induction on n that if $\mathfrak{M}', w' \models \chi^n_{[\mathfrak{M},w]}$ then player **II** has a wining strategy in the n-round game from $(\mathfrak{M}, w; \mathfrak{M}', w')$.

For $n = 0$ the claim follows by definition. Assuming it holds for n, let us look again at $\chi^{n+1}_{[\mathfrak{M},w]}$ from the perspective of games:

$$\chi^{n+1}_{[\mathfrak{M},w]} = \chi^0_{[\mathfrak{M},w]} \wedge \underbrace{\bigwedge_{(w,s)\in R} \Diamond \chi^n_{[\mathfrak{M},s]}}_{\text{forth}} \wedge \underbrace{\Box \bigvee_{(w,s)\in R} \chi^n_{[\mathfrak{M},s]}}_{\text{back}} .$$

The conjunct $\chi^0_{[\mathfrak{M},w]}$ guarantees that the game is not lost already.

The back-and-forth conjuncts tell Player **II** how to provide suitable responses in the first round to challenges from Player **I** played respectively in $\mathfrak{M}$ (forth) or in $\mathfrak{M}'$ (back).

Conversely, a failure of $\mathfrak{M}', w'$ to satisfy $\chi^n_{[\mathfrak{M},w]}$ gives Player **I** a winning strategy within n rounds.

Bisimulations, bisimulation games, characteristic formulae, and ML-equivalence: linking them all together

Theorem

For every pointed Kripke structures $(\mathfrak{M}, w)$ and $(\mathfrak{M}', w')$ of finite type the following are equivalent:

1. $\mathfrak{M}', w' \models \chi^n_{[\mathfrak{M}, w]}$.
2. $(\mathfrak{M}, w) \equiv_{\mathrm{ML}^n} (\mathfrak{M}', w')$.
3. $(\mathfrak{M}, w) \overset{n}{\rightleftarrows} (\mathfrak{M}', w')$.
4. Player **II** has a winning strategy in the n-round bisimulation game on $(\mathfrak{M}, w; \mathfrak{M}', w')$.

Proof:

$1 \Leftrightarrow 4$: already proved.

$3 \Leftrightarrow 4$: already proved.

$3 \Rightarrow 2$: already proved.

$2 \Rightarrow 1$: already proved (note that $\chi^n_{[\mathfrak{M}, w]} \in \mathrm{ML}^n$)