

Model Theory of Modal Logic

Lecture 5

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Bisimulations, bisimulation games, characteristic formulae, and ML-equivalence: linking them all together

Theorem

For every pointed Kripke structures (\mathfrak{M}, w) and (\mathfrak{M}', w') of finite type the following are equivalent:

1. $\mathfrak{M}', w' \models \chi_{[\mathfrak{M}, w]}^n$.
2. $(\mathfrak{M}, w) \equiv_{\text{ML}^n} (\mathfrak{M}', w')$.
3. $(\mathfrak{M}, w) \stackrel{n}{\leftrightarrow} (\mathfrak{M}', w')$.
4. Player **II** has a winning strategy in the n -round bisimulation game on $(\mathfrak{M}, w; \mathfrak{M}', w')$.

Corollary

For every pointed Kripke structures (\mathfrak{M}, w) and (\mathfrak{M}', w') of finite type the following are equivalent:

1. $\mathfrak{M}', w' \models \chi_{[\mathfrak{M}, w]}^n$ for every $n \in \mathbb{N}$.
2. $(\mathfrak{M}, w) \equiv_{ML} (\mathfrak{M}', w')$.
3. $(\mathfrak{M}, w) \rightleftarrows_f (\mathfrak{M}', w')$.
4. For every $n \in \mathbb{N}$ player **II** has a winning strategy in the n -round bisimulation game on $(\mathfrak{M}, w; \mathfrak{M}', w')$.

Corollary

Over Kripke structures of any finite type, finite bisimulation equivalence coincides with modal equivalence.

Corollary

Any formula $\varphi \in ML_n$ is logically equivalent to the disjunction $\bigvee_{\mathfrak{M}, w \models \varphi} \chi_{[\mathfrak{M}, w]}^n$.

Finite tree model property of modal logic

Lemma (finite tree model property)

For every $n \in \mathbb{N}$, every pointed Kripke structure (\mathfrak{M}, u) of a finite type is n -bisimilar to a finite tree structure. Consequently, any satisfiable formula of ML is satisfied at the root of a finite tree.

Proof sketch:

The unfolding $\mathfrak{M}[u]$ of (\mathfrak{M}, u) provides a bisimilar tree structure. As we only need n -bisimulation equivalence, we may cut off $\mathfrak{M}[u]$ at depth n from its root u , to obtain a tree structure $(\mathfrak{M}^n[u], u) \Leftrightarrow_n (\mathfrak{M}, u)$ whose depth is bounded by n .

This tree structure may still be infinite, due to infinite branching. In that case, however, we may prune successors at every node to retain at most one representative of each \Leftrightarrow_n equivalence class, i.e. for each characteristic formula of depth n .

As \Leftrightarrow_n has finite index the resulting tree structure is finite.

Classes closed under bisimulation

A class \mathcal{C} of pointed Kripke structures is **closed under bisimulation** if, whenever $(\mathfrak{M}, w) \in \mathcal{C}$ and $(\mathfrak{M}, w) \rightleftharpoons (\mathfrak{M}', w')$ then $(\mathfrak{M}', w') \in \mathcal{C}$.

Classes closed under n -bisimulations and finite bisimulation equivalence are defined likewise.

Corollary

Any class \mathcal{C} of pointed Kripke structures of a finite type that is closed under n -bisimulation is definable in ML^n by the disjunction $\bigvee_{(\mathfrak{M}, w) \in \mathcal{C}} \chi_{[\mathfrak{M}, w]}^n$.

Proposition

Let (\mathfrak{M}_1, w_1) and (\mathfrak{M}_2, w_2) be finite pointed Kripke structures with respectively n_1 and n_2 states, such that $(\mathfrak{M}_1, w_1) \rightleftharpoons_{n_1 n_2} (\mathfrak{M}_2, w_2)$. Then $(\mathfrak{M}_1, w_1) \rightleftharpoons (\mathfrak{M}_2, w_2)$.

Proof: Exercise.

Bisimulations between finite systems

Corollary

Let (\mathfrak{M}, w) and (\mathfrak{M}', w') be finite pointed Kripke structures with respectively n_1 and n_2 states, for which any of the equivalent conditions in the theorem holds for some $n \geq n_1 n_2$.

Then $(\mathfrak{M}, w) \rightleftharpoons (\mathfrak{M}', w')$.

Thus, within a finite Kripke structures \mathfrak{M} every state w can be characterized up to bisimulation equivalence by the characteristic formula $\chi_{[\mathfrak{M}, w]}^n$ for any large enough n .

We denote by $\chi_{[\mathfrak{M}, w]}$ the formula for the least suitable n .

Corollary

For every finite pointed Kripke structures (\mathfrak{M}, w) and a state $s \in \mathfrak{M}$ the following are equivalent:

1. $\mathfrak{M}, s \models \chi_{[\mathfrak{M}, w]}$.
2. $(\mathfrak{M}, w) \rightleftharpoons (\mathfrak{M}, s)$.

Finite vs full bisimulation equivalence

A frame is **finitely branching** if every state in it has only finitely many immediate successors.

Theorem (Hennessy–Milner theorem)

Let \mathfrak{M} and \mathfrak{M}' both be finitely branching KS.

Then $(\mathfrak{M}, w) \Leftrightarrow_f (\mathfrak{M}', w')$ implies $(\mathfrak{M}, w) \Leftrightarrow (\mathfrak{M}', w')$.

Proof sketch: Player **II** can maintain $(\mathfrak{M}, s) \Leftrightarrow_f (\mathfrak{M}', s')$ indefinitely in the bisimulation game starting from $(\mathfrak{M}, w; \mathfrak{M}', w')$, which gives her a winning strategy for the infinite game.

Indeed, let player **I** play in \mathfrak{M} and move the pebble from w to s . Suppose that for all responses s' available to player **II** in \mathfrak{M}' , $(\mathfrak{M}, s) \not\equiv_f (\mathfrak{M}', s')$. As there are only finitely many choices for s' there is a sufficiently large $n \in \mathbb{N}$ such that $(\mathfrak{M}, s) \not\equiv_n (\mathfrak{M}', s')$ for all s' with $(w', s') \in R'$. But then $(\mathfrak{M}, w) \not\equiv_{n+1} (\mathfrak{M}', w')$, which contradicts the assumption $(\mathfrak{M}, w) \Leftrightarrow_f (\mathfrak{M}', w')$.

Finite vs full bisimulation: Hennessy–Milner property

Corollary

Over finitely branching Kripke structures, modal equivalence coincides with bisimulation equivalence.

In general, the claim above does not hold.

Exercise: give an example.

A class of \mathcal{C} of Kripke structures **has the Hennessy–Milner property** if finite bisimulation equivalence (or, modal equivalence) of any two structures in \mathcal{C} implies bisimulation equivalence.

Thus, the class of finitely branching Kripke structures has the Hennessy–Milner property.

Infinitary modal logic

ML_∞ : the extension of the modal logic ML with infinite conjunctions and disjunctions, by adding the following clause to the definition of formulae:

If Ψ is any set of formulae of ML_∞ , then $\bigwedge \Psi$ and $\bigvee \Psi$ are formulae of ML_∞ .

The formulae of ML_∞ have an ordinal-valued nesting depth, because the nesting depth of an infinitary conjunction or disjunction is the supremum of the nesting depths of the constituent formulae.

The semantics of the infinite conjunctions and disjunctions is natural; e.g.:

$(M, w) \models \bigvee \Psi$ iff $(M, w) \models \psi$ for some $\psi \in \Psi$.

Bisimulation equivalence and infinitary modal equivalence

Theorem (Karp's theorem for modal logic)

Let (\mathfrak{M}, w) and (\mathfrak{M}', w') be Kripke structures of the same type.
Then the following are equivalent:

- (i) $(\mathfrak{M}, w) \rightleftharpoons (\mathfrak{M}', w')$.
- (ii) **II** has a winning strategy in the infinite bisimulation game from $(\mathfrak{M}, w; \mathfrak{M}', w')$.
- (iii) $(\mathfrak{M}, w) \equiv_{ML_\infty} (\mathfrak{M}', w')$.

Proof sketch:

- (i) \Leftrightarrow (ii): proved earlier.
- (ii) \Rightarrow (iii): if (\mathfrak{M}, w) is distinguished from (\mathfrak{M}', w') by a formula of nesting depth α , then Player **I** has a move forcing a successor configuration which is distinguished by a formula of nesting depth $\beta < \alpha$. By well-foundedness this gives Player **I** a winning strategy.
- (iii) \Rightarrow (ii): note that Player **II** can maintain ML_∞ equivalence indefinitely.

ω -saturated models

Let L be a FO language, \mathfrak{M} be an L -structure with domain W , and $A \subseteq W$. Then L_A is the extension of L with a constant name for each element of A , and \mathfrak{M}_A is the corresponding expansion of \mathfrak{M} .

Definition

An **element type with parameters in A** in L_A is a set Σ of L_A -formulae in a single free variable x .

The type Σ is a **type of \mathfrak{M}_A** if it is finitely consistent with the FO theory of \mathfrak{M}_A , that is $\mathfrak{M}_A \models \exists x \bigwedge \Sigma_0$ for every finite $\Sigma_0 \subseteq \Sigma$.

The type Σ is **realised** in \mathfrak{M}_A if $\mathfrak{M}_A, w \models \Sigma$ for some element w .

A structure \mathfrak{M} is **ω -saturated** if for every finite subset A every type of \mathfrak{M}_A is realised in \mathfrak{M}_A .

Theorem

Every FO structure has an ω -saturated elementary extension.

Theorem

The class of ω -saturated Kripke structures has the Hennessy–Milner property.

Bisimulation invariant first-order formulae

Definition (Bisimulation invariance of a FO-formula)

A formula $\gamma(x) \in \text{FO}_\tau$ of one free variable x is **bisimulation invariant** if for every pointed Kripke structures (\mathfrak{M}, u) and (\mathfrak{M}', u') , if $(\mathfrak{M}, u) \rightleftharpoons (\mathfrak{M}', u')$ then

$$\mathfrak{M}, u \models \gamma(x)[x := u] \text{ iff } \mathfrak{M}', u' \models \gamma(x)[x := u'].$$

Lemma

If $\gamma(x) \in \text{FO}$ is bisimulation invariant, then it is invariant under n -bisimulation for some $n \in \mathbb{N}$.

Proof sketch:

The one implication is immediate.

For the other, suppose that γ is not invariant under n -bisimulation for any $n \in \mathbb{N}$, and hence not equivalent to any modal formula.

Enumerate all modal formulae of the appropriate type as $(\psi_i)_{i \in \mathbb{N}}$.

Successively choose one of ψ_i or $\neg\psi_i$ to obtain a maximally consistent set T of modal formulae consistent with both γ and $\neg\gamma$.

Then there are pointed Kripke structures (\mathfrak{M}, w) and (\mathfrak{M}', w') such that both satisfy T , while

$\mathfrak{M}, w \models \gamma(x)[x := w]$ and $\mathfrak{M}', w' \models \neg\gamma(x)[x := w']$.

As (\mathfrak{M}, w) and (\mathfrak{M}', w') satisfy the same complete modal theory, $(\mathfrak{M}, w) \equiv_{\text{ML}} (\mathfrak{M}', w')$ and therefore $(\mathfrak{M}, w) \rightleftharpoons_f (\mathfrak{M}', w')$.

Passage to ω -saturated (or modally saturated) elementary extensions of (\mathfrak{M}, w) and (\mathfrak{M}', w') would then give structures $(\tilde{\mathfrak{M}}, w) \rightleftharpoons (\tilde{\mathfrak{M}}', w')$ which are still distinguished by γ , contradicting bisimulation invariance of γ .

ML is **the** bisimulation invariant fragment of FO

Theorem (van Benthem's modal characterisation theorem)

Let $\gamma(x) \in FO_\tau$. Then, the following are equivalent:

- (i) γ is bisimulation invariant.
- (ii) γ is logically equivalent to a formula $\tilde{\gamma} \in ML$.

Proof sketch:

(i) \Rightarrow (ii): proved earlier.

(i) \Rightarrow (ii): follows from the lemma, as any n -bisimulation invariant property is definable in ML_n .

Indeed, γ is then equivalent to a disjunction of characteristic formulae for n -bisimulation equivalence classes.

Zooming out: comparing ML and FO as logical languages

FO and ML are alternative languages to describe properties of Kripke structures and frames.

On a level of Kripke structures, ML is a fragment of FO²; in fact, it is precisely the bisimulation invariant fragment of FO².

On Kripke frames, however, ML is generally incompatible with FO.

On the one hand, there are very simple FO-definable properties of frames, such as irreflexivity, that are not definable by modal formulae.

On the other hand, there are non-first-order definable properties of Kripke frames, that are modally definable.

For instance, Gödel-Löb formula $\Box(\Box p \rightarrow p) \rightarrow \Box p$ defines the class of frames $\mathfrak{F} = (W, R)$ in which the relation R is transitive and inversely well-founded, i.e. there are no infinite R -chains.

Correspondence theory

Started in the 1970's, with works of S. Kripke, R. Goldblatt, S. Thomason, K. Fine, K. Segerberg, H. Sahlqvist, J. van Benthem (who coined the term 'correspondence theory'), and others, exploring systematically the questions:

Modal definability: What (first-order) frame properties are definable by modal formulae?

First-order definability: What modal formulae are first-order definable, i.e. define first-order properties on frames?

Both questions turned out to be algorithmically undecidable, but several model-theoretic characterizations or sufficient syntactic conditions have been identified (see the ML handbook chapter).

Modal logics defined semantically

A 'modal logic' can be defined semantically, as the modal theory of a class of Kripke frames, e.g.:

- The logic **K** of all Kripke frames.
- The logic **D** of all serial Kripke frames.
- The logic **T** of all reflexive Kripke frames.
- The logic **B** of all symmetric Kripke frames.
- The logic **K4** of all transitive Kripke frames.
- The logic **S4** of all reflexive and transitive Kripke frames.
- The logic **S5** of all reflexive, transitive, and symmetric Kripke frames, i.e. all equivalence relations.

The basic modal logic K as a deductive system

A sound and complete axiomatic system for deriving all valid modal formulae is the basic modal logic \mathbf{K} obtained by extending an axiomatic system for classical propositional logic with the axiom

$$K : \quad \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$$

and the Necessitation rule:

$$\frac{\varphi}{\Box \varphi}$$

Modal logics defined axiomatically

Many (but not all!) modal logics can be defined syntactically, as deductive systems, by extending **K** with additional axioms defining the respective class of Kripke frames.

- **T** = **K** + $\Box p \rightarrow p$;
- **D** = **K** + $\Box p \rightarrow \Diamond p$;
- **B** = **K** + $p \rightarrow \Box \Diamond p$;
- **K4** = **K** + $\Box p \rightarrow \Box \Box p$;
- **S4** = **T4** = **K4** + $\Box p \rightarrow p$;
- **S5** = **BS4** = **S4** + $p \rightarrow \Box \Diamond p$.

Completeness of modal logics

An axiomatically defined modal logic $L = K + Ax$, where Ax is a (usually finite) set of axioms is (Kripke)-complete iff for every modal formula φ :

$$L \vdash \varphi \text{ iff } FR(Ax) \models \varphi.$$

All axiomatic systems listed on the previous slide are complete with respect to validity in their respective classes of Kripke frames.

Unlike first-order logic, where Gödel's completeness theorem holds, **not every axiomatically defined modal logic is complete.**

For examples, see e.g. the book by Blackburn, de Rijke and Venema, or the Handbook of ML.

The problem of proving completeness of a given system of modal axioms has been one of the main problems driving the development of modal logic since the 1970's. Starting with Kripke's original papers, hundreds of completeness results have been obtained so far.

Canonicity

A most common technique for proving completeness is based on the **canonical model** technique, building models for satisfiable formulae from maximal consistent sets, à la Henkin.

However, that method does not work for all modal formulae.

A modal formula which is valid in its canonical frame, and therefore axiomatizes completely the validity in the class of frames which it defines is called **canonicity**.

Canonicity, first-order definability, and completeness are closely related by **Fine's theorem**: every complete and first-order definable modal formula is canonical.

Elementary and canonical modal formulae

Thus, the question of identifying first-order definable (aka, elementary) and canonical modal formulae is very important and has been studied in depth.

The first general results on such formulae are due to Lemmon and Scott, followed by a very general result by Sahlqvist (and, independently by van Benthem), in early 1970s, identifying a large syntactically defined class of elementary and canonical modal formulae, later called **Sahlqvist formulae**.

The class of Sahlqvist formulae has recently been substantially extended to the class of **inductive formulae** [VG and Vakarelov, APAL, 2006].

Furthermore, an algorithm (SQEMA) for identifying elementary and canonical modal formulae, and computing their FO equivalents, has been subsequently developed [Conradie, VG, and Vakarelov, LMCS, 2006] and implemented online on:

<http://www.fmi.uni-sofia.bg/fmi/logic/sqema>

Capita selecta of topics not covered in the course

- General frames and modal algebras.
- Ultrafilter extensions.
- Duality between algebraic semantics and Kripke semantics.
- Modal logic as a guarded fragment of first-order logic.
- Model theory of variations and extensions of modal logic: temporal logics, dynamic logics, logics of computations, modal μ -calculus, etc.
- Finite model theory of modal logic.

For all these, and more – see the ML handbook chapter.

Modal logic beyond model theory

Model theory of modal logic is a rich and exciting topic.

But, there is much more in modal logic than model theory, e.g.:

- Deductive modal logic.
- Computational modal logic.
- Applications: to mathematics, computer science, AI, linguistics, philosophy, etc.

For all these, and more – see the Handbook of Modal Logic.

Three concluding points

1. Modal logic is useful and important.
2. Moreover, modal logic is interesting.
3. Above all, modal logic is fun!

THE END!